CONSTRAINED MAXIMA MINIMA

Maxima and minima by Langrange's Multipliers Method.

Procedure

Step 1: Write $f(x,y,z)$ and $g(x,y,z)$

step 2: Find $F = f + \lambda g$

step 3: Find $\left.\begin{array}{l} \frac{\partial F}{\partial x} = 0 \\ \frac{\partial F}{\partial y} = 0 \\ \frac{\partial F}{\partial z} = 0 \end{array}\right\}$ $\frac{\partial F}{\partial \lambda} = 0$ → Find values of x, y, z

step 4: Substitute the value of x, y, z in $f(x,y,z)$

① Find the maximum value of $x \cdot y\, z$ subject to the condition $x + y + z = a$

$$f(x,y,z) = xyz$$

$$g(x,y,z) = x + y + z - a$$

$$F = f + \lambda g$$

$$F = xyz + \lambda(x + y + z - a) \rightarrow ①$$

$$\frac{\partial F}{\partial x} = yz + \lambda x$$

$$\frac{\partial F}{\partial y} = xz + \lambda y$$

$$\frac{\partial F}{\partial z} = xy + \lambda z$$

$$\frac{\partial F}{\partial \lambda} = x + y + z - a$$

$\frac{\partial F}{\partial x} = 0 \Rightarrow yz + \lambda = 0$

$-\lambda = yz \rightarrow ②$

$\frac{\partial F}{\partial y} = 0 \Rightarrow \cancel{y}z + \lambda = 0$

$-\lambda = xz \rightarrow ③$

$\frac{\partial F}{\partial y} = 0 \Rightarrow xy + \lambda = 0$

$-\lambda = xy \rightarrow ④$

$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow \quad x + y + z - a = 0$

$$x + y + z = a \rightarrow (5)$$

(2) & (3)

$$yz = xz$$

$$\boxed{y = x}$$

(3) & (4)

$$xz = xy$$

$$\boxed{z = y}$$

$$\boxed{x = y = z}$$

(5) $\Rightarrow$

$$x + x + x = a$$

$$3x = a$$

$$x = a/3, \quad y = a/3, \quad z = a/3$$

Maximum value is

$$f = xyz = a/3 \, a/3 \, a/3 = a^3/27$$

(2) A Rectangular box open at the top is to have a volume of 32cc. Find the dimension of the box requiring the least material for the construction

Sol $\quad$ ~~f(x, y, z)~~ z

Let x, y, z be the length, breadth & height of the box, the surface area of the box should be least.

The surface are $S = xy + 2yz + 2zx$. subject to $xyz = 32$

The auxiliary fn is $F = f + \lambda g$

$$F = (xy + 2yz + 2zx) + \lambda(xyz - 32) \rightarrow (1)$$

$$\frac{\partial F}{\partial x} = y + 2z + \lambda yz$$

$$\frac{\partial F}{\partial y} = x + 2z + \lambda z x$$

$$\frac{\partial F}{\partial z} = 2x + 2y + \lambda x y$$

$$\frac{\partial F}{\partial \lambda} = xyz - 32$$

$\frac{\partial F}{\partial x} = 0 \Rightarrow$

$$y + 2z + \lambda y z = 0$$

$$-\lambda y z = y + 2z$$

$$-\lambda = \frac{1}{z} + \frac{2}{y} \rightarrow (2)$$

$\frac{\partial F}{\partial y} = 0 \Rightarrow \quad x + 2z + \lambda z x = 0$

$$-\lambda z x = x + 2z$$

$$-\lambda = \frac{1}{z} + \frac{2}{x} \rightarrow (3)$$

$\frac{\partial F}{\partial z} = 0 \Rightarrow \quad 2x + 2y + \lambda x y = 0$

$$-\lambda = \frac{2}{y} + \frac{2}{x} \rightarrow (4)$$

$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow \quad xyz - 32 = 0 \rightarrow (5)$

(2) & (3) $\Rightarrow$ $\cancel{\frac{1}{z}} + \frac{2}{y} = \cancel{\frac{1}{z}} + \frac{2}{x}$

$$2x = 2y$$

$$x = y$$

(3) & (4) $\Rightarrow$ $\frac{1}{z} + \cancel{\frac{2}{x}} = \frac{2}{y} + \cancel{\frac{2}{x}}$

$x = y = 2z \qquad y = 2z$

(5) $\Rightarrow$ $(x)(x)(x/2) = 32$

$$x^3 = 64$$

$$x = 4$$

$\therefore x = 4,\ y = 4\ \&\ z = 2$

$\therefore$ Least material for the construction is

$$f = 16 + 2(4)(2) + 2(2)(4)$$

$$= 16 + 16 + 16 = 48$$

Find the minimum value of $x^2+y^2+z^2$ subject to the condition $x+y+z=3a$

$$f(x,y,z)=x^2+y^2+z^2$$

$$g(x,y,z)=x+y+z-3a$$

$$F=f+\lambda g$$

$$=x^2+y^2+z^2+\lambda(x+y+z-3a)$$

$$\frac{\partial F}{\partial x}=2x+\lambda \qquad \frac{\partial F}{\partial y}=2y+\lambda \qquad \frac{\partial F}{\partial z}=2z+\lambda$$

$\frac{\partial F}{\partial x}=0,\quad 2x+\lambda=0 \qquad \frac{\partial F}{\partial y}=0,\ 2y+\lambda=0 \qquad \frac{\partial F}{\partial z}=0 \quad . \ 2z+\lambda=0$

$-\lambda=2x \rightarrow (2) \qquad -\lambda=2y \rightarrow (3) \qquad -\lambda=2z \rightarrow (4)$

$\frac{\partial F}{\partial \lambda}=0 \Rightarrow \frac{\partial F}{\partial \lambda}=x+y+z-3a \quad , \quad x+y+z-3a=0 \rightarrow (5)$

From (2) & (3) $\quad 2x=2y \Rightarrow x=y$

$\Rightarrow x=y=z$

From (3) & (4) $\quad 2y=2z \Rightarrow y=z$

From (5) $\quad x+\cancel{x}+x=3a$

$x=a \Rightarrow y=a \ , \ z=a$

Minimum value is $f=x^2+y^2+z^2$

$=a^2+a^2+a^2=3a^2$

The temperature T at any point in space is $T=400xyz^2$ Find the heighest temperature in the surface of unit sphere $x^2+y^2+z^2=1$