Method of Variation of parameter

Steps to find the solution:-

Step 1: Let the eqn. equation be

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = X \quad \text{--- (1)}$$

Step 2: Find the CF of the diff. eqn. (1)

$$CF = A\,y_1(x) + B\,y_2(x) \quad \text{where } A \,\&\, B \text{ are constants}$$

Step 3: Find $y_1'(x)$, $y_2'(x)$ from $y_1(x)$ & $y_2(x)$

Step 4: Find Wronskian $W = y_1 y_2' - y_1' y_2$

Step 5: $P.I = u(x)\,y_1(x) + v(x)\,y_2(x)$

where $u(x) = -\int \frac{X y_2}{W}\,dx$

$v(x) = \int \frac{X y_1}{W}\,dx$

Step 6: The soln. is $y = C.F + P.I$

(1) Solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ by using method of variation of parameters.

Soln:

$(D^2+1)y = \operatorname{cosec} x$.

$\Rightarrow m^2+1=0$

$\Rightarrow m = \pm i$

CF is $y = A\cos x + B\sin x$.

$y_1(x) = \cos x \Rightarrow y_1'(x) = -\sin x$

$y_2(x) = \sin x \Rightarrow y_2'(x) = \cos x$

Wronskian $W = y_1y_2' - y_2y_1'$

$= \cos x \cdot \cos x - \sin x(-\sin x)$

$= \cos^2 x + \sin^2 x$

$= 1$

$P.I = u(x)y_1(x) + v(x)y_2(x)$

$u(x) = -\int \frac{X y_2}{W}\,dx$

$= -\int \frac{\operatorname{cosec} x \cdot \sin x\,dx}{1}$

$= -\int dx = -x$

$$v(x) = \int \frac{x y_1}{W} dx$$

$$= \int \frac{\operatorname{cosec} x \cos x}{1} dx$$

$$= \int \cot x \, dx$$

$$= \log(\sin x)$$

$$P.I = -x\cos x + \sin x \log(\sin x)$$

$\therefore$ Soln. is $y = A\cos x + B\sin x - x\cos x + \sin x \log(\sin x)$

② $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$.

$$(D^2+4)y = 0$$

$$\Rightarrow m^2+4 = 0$$

$$\Rightarrow m = \pm 2i$$

C.F. is $y = A\cos 2x + B\sin 2x$

$y_1(x) = \cos 2x \Rightarrow y_1'(x) = -2\sin 2x$

$y_2(x) = \sin 2x \Rightarrow y_2'(x) = 2\cos 2x$

$$W = \cos 2x(2\cos 2x) - \sin 2x(-2\sin 2x)$$

$$= 2\cos^2 2x + 2\sin^2 2x$$

$$= 2$$

$$P.I = u(x)y_1(x) + v(x)y_2(x)$$

$$u(x) = -\int \frac{x y_2}{W} dx$$

$$= -\int \frac{4\tan 2x}{2}\sin 2x\,dx$$

$$= -2\int \frac{\sin^2 2x}{\cos 2x}\,dx$$

$$= -2\int \frac{(1-\cos^2 2x)}{\cos 2x}\,dx$$

$$= -2\left[\int \sec 2x\,dx - \int \cos 2x\,dx\right]$$

$$= -2\left[\frac{1}{2}\log(\sec 2x+\tan 2x) - \frac{\sin 2x}{2}\right]$$

$$= \sin 2x - \log(\sec 2x+\tan 2x)$$

$$v(x) = \int \frac{\cos 2x}{2}\cdot 4\tan 2x\,dx.$$

$$= 2\int \sin 2x\,dx$$

$$= -\cos 2x$$

$$P.I = \left[\sin 2x - \log(\sec 2x+\tan 2x)\right]\cos 2x + $$
$$-\cos 2x\sin 2x$$

$$= -\cos 2x\log(\sec 2x+\tan 2x)$$

Soln. is $y = A\cos 2x + B\sin 2x - \cos 2x\log(\sec 2x+\tan 2x)$

3) Solve $y''+y=\sec^2 x$ by the method of variation of parameter

Soln: $(D^2+1)y=\sec^2 x$

AE is $m^2+1=0$

$\Rightarrow m=\pm i$

C.F. $y=A\cos x+B\sin x$

$y_1(x)=\cos x \Rightarrow y_1'(x)=-\sin x$

$y_2(x)=\sin x \Rightarrow y_2'(x)=\cos x.$

$W=\cos^2 x+\sin^2 x$

$=1$

$P.I.=u(x)\,y_1(x)+v(x)\,y_2(x)$

$u(x)=-\int \frac{X\,y_2(x)}{W}\,dx$

$=-\int \sec^2 x\,\sin x\,dx$

$=-\int \sec x\tan x\,dx$

$=-\int d(\sec x)$

$u(x)=-\sec x$

$$v(x) = \int \frac{x\, y_1(x)}{w}\, dx$$

$$= \int \sec^2 x \cos x\, dx$$

$$= \int \sec x\, dx$$

$$= \log(\sec x + \tan x)$$

$$P.I_1 = -\sec x \cos x + \sin x \log(\sec x + \tan x)$$

$$= -1 + \sin x \log(\sec x + \tan x)$$

$\therefore$ Soln. is $y = A\cos x + B\sin x - \sin x \log(\sec x + \tan x)$

5) $\frac{d^2y}{dx^2} + y = \cot x$.

Soln: $y = A\cos x + B\sin x - \sin x \log(\operatorname{cosec} x + \cot x)$