



DEPARTMENT OF MATHEMATICS

UNIT - V SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

Method of Variation of parameter
Steps to find the solution:-

Step 1: Let the eqn. equation be

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = x \quad \text{--- (1)}$$

Step 2: Find the CF of the diff. eqn. (1)

$$CF = Ay_1(x) + By_2(x) \text{ where } A \& B \text{ are constants.}$$

Step 3: Find $y_1'(x), y_2'(x)$ from $y_1(x) \& y_2(x)$

Step 4: Find Wronskian $W = y_1 y_2' - y_1' y_2$

Step 5: P.I = $u(x)y_1(x) + v(x)y_2(x)$

$$\text{where } u(x) = - \int \frac{x y_2}{W} dx$$

$$v(x) = \int \frac{x y_1}{W} dx$$

Step 6: The soln. is $y = C.F. + P.I.$

$\int n^2 dx = \frac{n^3}{3}$
 $\int \frac{1}{n} dx = \ln n$
 $\int \tan dx = \ln |\sec x|$
 $\int \cot dx = \ln |\sin x|$
 $\int \sec dx = \ln |\sec x + \tan x|$
 $\int \csc dx = \ln |\csc x - \cot x|$



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① Solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ by using method of variation of parameters.

soln:
 $(D^2+1)y = \operatorname{cosec} x$

$$\Rightarrow m^2 + 1 = 0$$

$$\Rightarrow m = \pm i$$

CF is $y = A \cos x + B \sin x$

$$y_1(x) = \cos x \Rightarrow y_1'(x) = -\sin x$$

$$y_2(x) = \sin x \Rightarrow y_2'(x) = \cos x$$

$$\begin{aligned} \text{Wronskian } W &= y_1 y_2' - y_2 y_1' \\ &= \cos x \cdot \cos x - \sin x (-\sin x) \\ &= \cos^2 x + \sin^2 x \\ &= 1 \end{aligned}$$

$$P.S = u(x)y_1(x) + v(x)y_2(x)$$

$$\begin{aligned} u(x) &= - \int \frac{x y_2}{W} dx \\ &= - \int \frac{\operatorname{cosec} x \cdot \sin x}{1} dx \\ &= - \int dx = -x \end{aligned}$$



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$$\begin{aligned}v(x) &= \int \frac{x y_1}{w} dx \\&= \int \frac{\cos e^x \cos x}{1} dx \\&= \int \cot x dx \\&= \log(\sin x)\end{aligned}$$

$$P.I = -x \cos x + \sin x \log(\sin x)$$

$$\therefore \text{Soln. is } y = A \cos x + B \sin x - x \cos x + \sin x \log(\sin x)$$

$$\textcircled{2} \frac{d^2y}{dx^2} + 4y = 4 \tan 2x$$

$$(D^2 + 4)y = 0$$

$$\Rightarrow m^2 + 4 = 0$$

$$\Rightarrow m = \pm 2i$$

$$C.F. \text{ is } y = A \cos 2x + B \sin 2x$$

$$y_1(x) = \cos 2x \Rightarrow y_1'(x) = -2 \sin 2x$$

$$y_2(x) = \sin 2x \Rightarrow y_2'(x) = 2 \cos 2x$$

$$w = \cos 2x (2 \cos 2x) - \sin 2x (-2 \sin 2x)$$

$$= 2 \cos^2 2x + 2 \sin^2 2x$$

$$= 2$$

$$P.I = u(x)y_1(x) + v(x)y_2(x)$$

$$u(x) = - \int \frac{x y_2}{w} dx$$



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$$= - \int \frac{4 \tan 2x \sin 2x}{2} dx$$

$$= -2 \int \frac{\sin^2 2x}{\cos 2x} dx$$

$$= -2 \int \frac{(1 - \cos^2 2x)}{\cos 2x} dx$$

$$= -2 \left[\int \sec 2x dx - \int \cos 2x dx \right]$$

$$= -2 \left[\frac{1}{2} \log (\sec 2x + \tan 2x) - \frac{\sin 2x}{2} \right]$$

$$= \sin 2x - \log (\sec 2x + \tan 2x)$$

$$v(x) = \int \frac{\cos 2x \cdot 4 \tan 2x}{2} dx$$

$$= 2 \int \sin 2x dx$$

$$= -\cos 2x$$

$$PI = \left[\sin 2x - \log (\sec 2x + \tan 2x) \right] \cos 2x +$$
$$- \cos 2x \sin 2x$$

$$= -\cos 2x \log (\sec 2x + \tan 2x)$$

$$\text{soln. is } y = A \cos 2x + B \sin 2x - \cos 2x \log (\sec 2x + \tan 2x)$$



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3) Solve $y'' + y = \sec^2 x$ by the method of variation of parameter

Soln: $(D^2 + 1)y = \sec^2 x$

AE is $m^2 + 1 = 0$

$\Rightarrow m = \pm i$

C.F. $y = A \cos x + B \sin x$

$y_1(x) = \cos x \Rightarrow y_1'(x) = -\sin x$

$y_2(x) = \sin x \Rightarrow y_2'(x) = \cos x$

$W = \cos^2 x + \sin^2 x$

$= 1$

P.I. $= u(x)y_1(x) + v(x)y_2(x)$

$u(x) = -\int \frac{x y_2'(x)}{W} dx$

$= -\int \sec^2 x \sin x dx$

$= -\int \sec x \tan x dx$

$= -\int d(\sec x)$

$u(x) = -\sec x$



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$$\begin{aligned}v(x) &= \int \frac{x y_1(x)}{w} dx \\&= \int \sec^2 x \cos x dx \\&= \int \sec x dx \\&= \log(\sec x + \tan x)\end{aligned}$$

$$\begin{aligned}p \cdot I_1 &= -\sec x \cos x + \sin x \log(\sec x + \tan x) \\&= -1 + \sin x \log(\sec x + \tan x)\end{aligned}$$

$$\therefore \text{Soln. is } y = A \cos x + B \sin x - \sin x \log(\sec x + \tan x)$$

$$5) \frac{d^2y}{dx^2} + y = \cot x.$$

$$\text{Soln: } y = A \cos x + B \sin x - \sin x \log(\csc x + \cot x)$$