CAUCHY'S LINEAR EQUATION

The general form is

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = f(x)$$

where $a_1, a_2, \ldots, a_n$ are constants & $f(x)$ is a func. of x.

Let us take $x = e^z$

$$\Rightarrow \log x = z$$

(i.e) $z = \log x$.

By solving we get,

$$x \frac{dy}{dx} = xDy = D'y$$

$$x^2 \frac{d^2 y}{dx^2} = D'(D'-1)y$$

$$x^3 \frac{d^3 y}{dx^3} = D'(D'-1)(D'-2)y$$

① Solve: $x^2y'' + 2xy' + 2y = 0$

Soln: $(x^2D^2 + 2xD + 2)y = 0$

put $x = e^z \Rightarrow z = \log x$

$xD = D'$

$x^2D^2 = D'(D'-1)$

$\Rightarrow [D'(D'-1) + 2D' + 2]y = 0$

$\Rightarrow [D'^2 + D' + 2]y = 0$

A.E is $m^2 + m + 2 = 0$

$\Rightarrow m = \frac{-1 \pm \sqrt{7}i}{2}$

$y = e^{-\frac{1}{2}z}\left[A\cos\frac{\sqrt{7}}{2}z + B\sin\frac{\sqrt{7}}{2}z\right]$

$= e^{-\frac{1}{2}\log x}\left[A\cos\frac{\sqrt{7}}{2}\log x + B\sin\frac{\sqrt{7}}{2}\log x\right]$

$= e^{\log x^{-\frac{1}{2}}}\left[A\cos\frac{\sqrt{7}}{2}\log x + B\sin\frac{\sqrt{7}}{2}\log x\right]$

$= \frac{1}{\sqrt{x}}\left[A\cos\frac{\sqrt{7}}{2}\log x + B\sin\frac{\sqrt{7}}{2}\log x\right]$

(2) Solve: $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = x^2\cos(\log x)$

Soln: $(x^2D^2 - 3xD + 4)y = x^2\cos(\log x)$

put $x = e^z \Rightarrow z = \log x$

$xD = D'$

$x^2D^2 = D'(D'-1)$

$\Rightarrow [D'(D'-1) - 3D' + 4]y = e^{2z}\cos z.$

$\Rightarrow [D'^2 - 4D' + 4]y = e^{2z}\cos z.$

AE is $m^2 - 4m + 4 = 0$

$\Rightarrow m = 2, 2$

C.F is $y = (Az + B)e^{2z}$

$$P.I_1 = \frac{1}{(D'-2)^2}\cdot e^{2z}$$

$$= \frac{1}{D'^2 - 4D' + 4}\, e^{2z}$$

$$= \frac{1}{4 - 8 + 4}\, e^{2z}$$

$$= \frac{1}{0}\, e^{2z}$$

$$= \frac{1}{2D'-4} xe^{2x}$$

$$= \frac{1}{4-4} xe^{2x}$$

$$= \frac{1}{2} x^2 e^{2x}$$

$$P.I_2 = \frac{1}{D^2-4D'+4} \cos x$$

$$= \frac{1}{-1-4D'+4} \cos x$$

$$= \frac{1}{3-4D'} \cos x$$

$$= \frac{3+4D'}{9-16D'^2} \cos x$$

$$= \frac{3\cos x + 4(-\sin x)}{9-16(-1)}$$

$$= \frac{3\cos x - 4\sin x}{25}$$

$$C.S.\ y = (Ax+B)e^{2x} + \frac{x^2}{2}e^{2x} + \frac{3\cos x - 4\sin x}{25}$$

$$= (A\log x + B)e^{2\log x} + \frac{(\log x)^2}{2} e^{2\log x} + \frac{3\cos(\log x) - 4\sin(\log x)}{25}$$

$$= (A\log x + B)x^2 + \frac{(\log x)^2}{2} x^2 + \frac{1}{25}[3\cos(\log x) - 4\sin(\log$$