Method of Variation of Parameters:

This method is very useful in finding the general solution of the second order equation.

$$(D^2 + a_1 D + a_2) y = X$$

If $(D^2 + a_1 D + a_2) y = 0$, then CF is the G.S.

$$C.F = c_1 f_1 + c_2 f_2$$

$$P.I = P f_1 + Q f_2$$

where $P = \int \frac{f_2 X}{W} dx \quad Q = \int \frac{f_1 X}{W} dx.$

where W = Wronskian of f_1 & f_2

$$= \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

① $(D^2 + 4) y = \sec 2x$ (or) $\frac{d^2 y}{dx^2} + 4y = \sec 2x$.

solve using method of variation of parameters.

$\therefore$ Given eqn is $(D^2 + 4) y = \sec 2x$.

The auxiliary eqn is Replace D by m

$$(m^2 + 4) = 0.$$

$$m^2 = -4 \qquad 0 \pm 2i$$

$$\boxed{m = \pm 2i} \qquad \alpha = 0; \beta = \pm 2$$

$\therefore$ The roots are imaginary,

C.F is $e^{0x}(A \cos 2x + B \sin 2x)$

i.e., $A \cos 2x + B \sin 2x$.

From CF, $f_1 = \cos 2x$ | $f_2 = \sin 2x$

$f_1' = -\sin 2x \cdot 2$ | $f_2' = \cos 2x \cdot 2$

$$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= (\cos 2x)(2\cos 2x) - (\sin 2x)(-2\sin 2x)$$

$$= 2(\cos 2x)^2 + 2(\sin 2x)^2 = 2\left[\sin^2 2x + \cos^2 2x\right]$$

$$W = 2$$

$P.I = Pf_1 + Qf_2$

$P = -\int \frac{f_2 X}{W}$ ← RHS $\quad Q = \int \frac{f_1 X}{W}$ ↙ RHS

$= -\int \frac{(\sin 2x)(\sec 2x)}{2} dx$

$= -\int \frac{\sin 2x}{2\cos 2x} dx = \frac{1}{2}\int \tan 2x \, dx.$

$= \frac{-1}{2} \frac{\log(\sec 2x)}{2}$

$= \frac{-1}{4} \log(\sec 2x).$

$Q = \int \frac{f_1 X}{W} dx$

$= \int \frac{\cos 2x \sec 2x}{2} dx$

$= \frac{1}{2} \int \cos 2x \cdot \frac{1}{\cos 2x} dx$

$= \frac{1}{2}\int dx \quad Q = \frac{x}{2}.$

∴ The particular Integral is $PI = \frac{-1}{4} \log(\sec 2x)\cancel{(\sin 2x)}(\cos 2x)$

$+ \frac{x}{2} \sin 2x.$

$y = C.F + P.I$

$= A\cos 2x + B\sin 2x$

$-\frac{1}{4} \log(\sec 2x)(\cos 2x)$

$+ \frac{x}{2} \sin 2x$ //

2) Solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ using method of variation of parameters

Soln:

The given eqn can be written as

$(D^2+1)y = \operatorname{cosec} x$

The auxiliary eqn is

$m^2 + 1 = 0$

$m^2 = -1$

$m = \pm 1i$

$\alpha \pm i\beta = 0 \pm i$

$\alpha = 0$ and $\beta = 1$

∴ The complementary function is

$e^{0x}(A\cos x + B\sin x).$

$\Rightarrow C.F = A\cos x + B\sin x.$

$f_1 = \cos x \quad | \quad f_2 = \sin x$

$f_1' = -\sin x \quad | \quad f_2' = +\cos x$

$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= \cos^2 x + \sin^2 x = 1$

$\therefore \boxed{W = 1}$

$P.I = Pf_1 + Qf_2$

$P = -\int \frac{f_2 X}{W}$ $Q = \int \frac{f_1 X}{W}$

$P = -\int \frac{(\sin x)(\operatorname{cosec} x)}{1} dx$

$= -\int (\sin x)\left(\frac{1}{\sin x}\right) dx$

$= -\int dx = -x.$

$\boxed{P = -x}$

$Q = \int \frac{f_1 X}{W} dx$

$= \int \frac{(\cos x)(\operatorname{cosec} x)}{1} dx$

$= \int \frac{\cos x}{\sin x} dx = \int \cot x\, dx$

$Q = \log(\sin x).$

$P.I = (-x)(\cos x)$

$+ \log(\sin x)(\sin x)$

$= -x\cos x + \sin x \log(\sin x)$

The general solution

$y = C.F + P.I$

$= A\cos x + B\sin x$

$-x\cos x + \sin x\log(\sin x)$

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3) $(D^2 + a^2)y = \sec(ax)$

The auxiliary eqn is

$m^2 + a^2 = 0$

$m^2 = -a^2$

$m = \pm ia$

The complementary function is $e^{0x}(A\cos ax + B\sin ax)$

C.F is $A\cos ax + B\sin ax$.

$f_1 = \cos ax$	$f_2 = \sin ax$
$f_1' = -\sin ax . a$	$f_2' = \cos ax . a$

$PI = Pf_1 + Qf_2$

$P = -\int \frac{f_2 X}{W} = -\int \frac{(\sin ax)(\sec ax)}{W}$

$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$

$= a\cos^2 ax + a\sin^2 ax$

$= a(\cos^2 ax + \sin^2 ax)$

$= a(1)$

$W = 1$

$P = -\int \frac{\sin ax}{\cos ax} = -\int \tan ax.$

$= -\log(\sec ax).$

$P = -\log(\sec ax).$

$Q = \int \frac{f_1 X}{W} = \int \frac{\cos ax . \sec ax}{1} dx$

$= \int dx = x$

$P.I = -\log(\sec ax) . \cos ax$

$+ x\sin ax$