Taylor Series Expansion of two variables

Given point (a, b)

$h = (x-a)$

$k = (y-b)$

Let $f(x,y)$ be a function of two variables x and y. We can expand $f(x+h, y+k)$ in a series of ascending powers of h and k.

~~$f(x+h, y+k)$~~

$$f(x,y) = f(a,b) + \left[h f_x(a,b) + k f_y(a,b)\right] + \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right] + \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right] + \cdots$$

Problem: 1.

Expand $e^x \cos y$ about $(0, \frac{\pi}{2})$, upto the third term using taylor's series.

Solution:

Given: $f(x,y) = e^x \cos y$

$(a,b) = (0, \pi/2)$

Function	Value at $(0, \pi/2)$
$f(x,y) = e^x \cos y$	$f = 0$
$f_x = e^x \cos y$	$f_x = 0$
$f_y = -e^x \sin y$	$f_y = -1$
$f_{xx} = e^x \cos y$	$f_{xx} = 0$
$f_{xy} = -e^x \sin y$	$f_{xy} = -1$
$f_{yy} = -e^x \cos y$	$f_{yy} = 0$
$f_{xxx} = e^x \cos y$	$f_{xxx} = 0$
$f_{xxy} = -e^x \sin y$	$f_{xxy} = -1$
$f_{xyy} = -e^x \cos y$	$f_{xyy} = 0$
$f_{yyy} = e^x \sin y$	$f_{yyy} = 1$

By Taylor's theorem,

$f(x,y) =$ formula

$\cancel{f(x,y)=}$ $\quad a=0\ ;\ b=\frac{\pi}{2}$

$$h = x-a = x-0 = x$$

$$k = y-b = y-\frac{\pi}{2}$$

$$f(x,y) = 0+\left[x(0)+\left(y-\frac{\pi}{2}\right)(-1)\right]$$

$$+\frac{1}{2!}\left[x^2(0)+2x\left(y-\frac{\pi}{2}\right)(-1)+\left(y-\frac{\pi}{2}\right)^2(0)\right]$$

$$+\frac{1}{3!}\left[x^3(0)+3x^2\left(y-\frac{\pi}{2}\right)(-1)+3x\left(y-\frac{\pi}{2}\right)^2(0)\right.$$

$$\left.+\left(y-\frac{\pi}{2}\right)^3(1)\right]+\cdots$$

$$=0+\left[0-y+\frac{\pi}{2}\right]+\frac{1}{2}\left[0-2xy+2x\frac{\pi}{2}+0\right]$$

$$+\frac{1}{6}\left[0-3x^2y+3x^2\frac{\pi}{2}+\left(y-\frac{\pi}{2}\right)^3\right]+\cdots$$

$$=-y+\frac{\pi}{2}+\frac{1}{2}(-2xy)+\frac{1}{2}\left(2x\frac{\pi}{2}\right)$$

$$+\frac{1}{6}(-3x^2y)+\frac{1}{6}\left(3x^2\frac{\pi}{2}\right)+\frac{1}{6}\left(y-\frac{\pi}{2}\right)^3+\cdots$$

$$=-y+\frac{\pi}{2}-xy+x\frac{\pi}{2}-\frac{x^2y}{2}+\frac{1}{2}x^2\frac{\pi}{2}+\cdots$$

$$f(x,y)\simeq -y-xy-\frac{x^2y}{2}+\frac{\pi}{2}+\frac{x\pi}{2}+\frac{x^2\pi}{4}+\cdots$$

2 H.W: $f(x,y)=e^x\sin y$ at point $\left[1,\frac{\pi}{2}\right]$

1 & 2 Redo with $[0,0]$

③ Expand $e^x \log(1+y)$ in powers of x and y upto terms of third degree.

Soln: $f(x,y) = e^x \log(1+y)$ point $(0,0)$ (a,b)

$h = x - a = x$; $k = y - b = y$.

Function	Value at $(0,0)$
$f(x,y) = e^x \log(1+y)$	$f = 0$
$f_x = e^x \log(1+y)$ $f_y = e^x (1+y)^{-1}$	$f_x = 0$ $f_y = 1$
$f_{xx} = e^x \log(1+y)$ $f_{xy} = e^x (1+y)^{-1}$ $f_{yy} = -e^x (1+y)^{-2}$	$f_{xx} = 0$ $f_{xy} = 1$ $f_{yy} = -1$
$f_{xxx} = e^x \log(1+y)$ $f_{xxy} = e^x (1+y)^{-1}$ $f_{xyy} = -e^x (1+y)^{-2}$ $f_{yyy} = 2e^x (1+y)^{-3}$.	$f_{xxx} = 0$ $f_{xxy} = 1$ $f_{xyy} = -1$ $f_{yyy} = 2$.

$$f(x,y) = 0 + [x(0) + y(1)] + \frac{1}{2!}[x^2(0) + 2xy(1) + y^2(-1)] + \frac{1}{3!}[x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)] + \cdots$$

$$= 0 + [0 + y] + \frac{1}{2}[0 + 2xy - y^2] + \frac{1}{6}[0 + 3x^2y - 3xy^2 + 2y^3] + \cdots$$

$$= y + \frac{1}{2}(2xy - y^2) + \frac{1}{6}(3x^2y - 3xy^2 + 2y^3) + \cdots$$