Total derivatives

If $u = f(x, y)$ is a function of x and y, where $x = f(t)$; $y = g(t)$ then.

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}.$$

$\frac{du}{dt} \rightarrow$ total derivative.

$\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y} \rightarrow$ partial derivatives.

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1) Find $\frac{du}{dt}$ if $u = x^3 y^4$ where $x = t^3$ and $y = t^2$.

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$x = t^3$, $\frac{dx}{dt} = 3t^2$ | $y = t^2$, $\frac{dy}{dt} = 2t$

$u = x^3 y^4$

$\frac{\partial u}{\partial x} = y^4 3x^2$

$\frac{\partial u}{\partial x} = 3x^2 y^4$

$\frac{\partial u}{\partial y} = x^3 4y^3$

$\frac{\partial u}{\partial y} = 4x^3 y^3$

$$\therefore \frac{du}{dt} = 3x^2y^4(3t^2) + 4x^3y^3(2t)$$

$$= 9x^2y^4t^2 + 8t x^3y^3$$

$$= 9(t^3)^2(t^2)^4t^2 + 8(t)(t^3)^3(t^2)^3$$

$$= 9t^6t^8t^2 + 8t^1t^9t^6$$

$$= 9t^{16} + 8t^{16}$$

$$= t^{16}(9+8) = 17t^{16}$$

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2) If $u = x^2y^3$, $x = \log t$; $y = e^t$ find $\frac{du}{dt}$

Soln:

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$x = \log t$, $\frac{dx}{dt} = \frac{1}{t}$ | $y = e^t$, $\frac{dy}{dt} = e^t$

$u = x^2y^3$

$\frac{\partial u}{\partial x} = y^3(2x) = 2xy^3$

$\frac{\partial u}{\partial y} = x^2(3y^2) = 3x^2y^2$

$\therefore \frac{du}{dt} = 2xy^3\left(\frac{1}{t}\right) + 3x^2y^2(e^t)$

$= 2(\log t)(e^t)^3\left(\frac{1}{t}\right) + 3(\log t)^2 (e^t)^2(e^t)$

$\frac{du}{dt} = \frac{2\log t\, e^{3t}}{t} + 3(\log t)^2 e^{3t}$

3. If $u = xy + yz + zx$ where $x = \frac{1}{t}$; $y = e^t$; $z = e^{-t}$. Find $\frac{du}{dt}$.

Soln:

3 variables

$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$

$u = xy + yz + zx$.

$\frac{\partial u}{\partial x} = y(1) + 0 + z(1)$

$= y + z$

$\frac{\partial u}{\partial y} = x(1) + z(1) + 0$

$= x + z$.

$\frac{\partial u}{\partial z} = 0 + y(1) + x(1)$

$= y + x$.

$x = \frac{1}{t}$	$y = e^t$	$z = e^{-t}$
$\frac{dx}{dt} = -\frac{1}{t^2}$	$\frac{dy}{dt} = e^t$	$\frac{dz}{dt} = -e^{-t}$

$\frac{du}{dt} = (y+z)\left(-\frac{1}{t^2}\right) + (x+z)(e^t) + (y+x)(-e^{-t})$

$= (e^t + e^{-t})\left(-\frac{1}{t^2}\right) + \left(\frac{1}{t} + e^{-t}\right)(e^t) + \left(e^t + \frac{1}{t}\right)(-e^{-t})$

$= -\frac{(e^t + e^{-t})}{t^2} + e^t\left(\frac{1}{t} + e^{-t}\right) - e^{-t}\left(e^t + \frac{1}{t}\right)$.

4) If $u = e^x \sin y$ where $x = st^2$ and $y = s^2 t$. find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$.

Soln.

$u = e^x \sin y$

$x = st^2$ and $y = s^2 t$

$u = e^{st^2}\sin(s^2 t)$

$\frac{\partial u}{\partial s} = e^{st^2}\cos(s^2 t)\, t(2s) + \sin(s^2 t)\cdot e^{st^2}\cdot(t^2)$

$= e^{st^2}\, 2st \cos s^2 t + t^2 e^{st^2}\sin(s^2 t)$

$= 2st\, e^{st^2}\cos(s^2 t)$

Implicit functions:

Let $f(x,y)=c$ and w.r.t $u=f(x,y)$. Then

$$\frac{dy}{dx}=-\frac{(\partial f/\partial x)}{(\partial f/\partial y)}$$

Example-1

Find $\frac{dy}{dx}$ if $x^3+y^3=3axy$.

Soln:

Given: $x^3+y^3-3axy=0$.

Now Let

$f(x,y)=x^3+y^3-3axy$

$$\frac{\partial f}{\partial x}=3x^2+0-3ay$$
$$=3(x^2-ay)$$

$$\frac{\partial f}{\partial y}=0+3y^2-3ax$$
$$=3(y^2-ax)$$

$$\frac{dy}{dx}=-\frac{(\partial f/\partial x)}{(\partial f/\partial y)}$$

$$=-\frac{(3(x^2-ay))}{(3(y^2-ax)}$$

$$\frac{dy}{dx}=\frac{-x^2+ay}{y^2-ax} \quad //$$

Example-2:

Find $\frac{dy}{dx}$ if $3x^2+xy-$ $+4x-2y+1=0$.

Soln:

Given: $3x^2+xy-y^2+4x-2y$

Let $f(x,y)=$

$3x^2+xy-y^2+4x-2y+1$

$$\frac{\partial f}{\partial x}=6x+y-0+4-0+0$$

$$\frac{\partial f}{\partial x}=6x+y+4$$

$$\frac{\partial f}{\partial y}=0+x-2y+0-2+0$$

$$\frac{\partial f}{\partial y}=x-2y-2$$

$$\frac{dy}{dx}=\frac{6x+y+4}{x-2y-2}$$

Example-3.

Find $\frac{du}{dx}$ if $u=\cos(x^2+y^2)$ where $a^2x^2+b^2y^2=c^2$.

Solution:

Total derivative

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

Here the ~~paramete~~ variab changed to x.

$$\therefore \frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$\therefore \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \quad \text{--- (1)}$$

Given: $u = \cos(x^2+y^2)$

$$\frac{\partial u}{\partial x} = -\sin(x^2+y^2)\,[2x] \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial y} = -\sin(x^2+y^2)\,[2y] \quad \text{--- (3)}$$

also gn:

$$a^2x^2 + b^2y^2 = c^2$$

Diff w.r.to. 'x' we get

$$a^2(2x) + b^2(2y)\cdot\frac{dy}{dx} = 0$$

$$2yb^2\frac{dy}{dx} = -2a^2x$$

$$\frac{dy}{dx} = \frac{-a^2x}{b^2y} \quad \text{--- (4)}$$

sub (2), (3), (4) in (1).

$$\frac{du}{dx} = -\sin(x^2+y^2)(2x) + \left(-\sin(x^2+y^2)(2y)\right)\cdot\left(\frac{-a^2x}{b^2y}\right)$$

$$= -2x\sin(x^2+y^2) + \frac{2a^2x}{b^2}\cdot\sin(x^2+y^2)$$

$$= 2\sin(x^2+y^2)\left[\frac{a^2x}{b^2} - x\right]$$

$$= 2x\sin(x^2+y^2)\left(\frac{a^2}{b^2} - 1\right)$$

Example 4:

If z is a function of x and y where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, show that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

Solution: $z = f(u,v)$ composite

Given: ~~$z = f(x,y)$ and~~

Then, total derivatives (check with the qn)

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{dx}{du} + \frac{\partial z}{\partial y}\cdot\frac{dy}{du}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dv} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dv}$$

$$\frac{\partial x}{\partial u} = e^u \quad \Big| \quad \frac{\partial x}{\partial v} = -e^{-v}$$

$$\frac{\partial y}{\partial u} = -e^{-u} \quad \Big| \quad \frac{\partial y}{\partial v} = -e^v$$

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot e^u + \frac{\partial z}{\partial y}\cdot(-e^{-u})$$

$$= \frac{\partial z}{\partial x}\cdot e^u - \frac{\partial z}{\partial y}\cdot e^{-u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}\cdot(-e^{-v}) + \frac{\partial z}{\partial y}\cdot(-e^{v})$$

$$= -e^{-v}\frac{\partial z}{\partial x} - e^{v}\frac{\partial z}{\partial y}.$$

LHS

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^{u} + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^{v})\frac{\partial z}{\partial y}$$

$$= x\cdot\frac{\partial z}{\partial x} - y\cdot\frac{\partial z}{\partial y} = RHS.$$

Hence proved.

Example-5:

If $u = \sin^{-1}(x-y)$,

$x = 3t$; $y = 4t^3$.

S.T. $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$.

Soln:

Note: x and y are in terms of t alone.

$\therefore$ diffn $x+y$ normally.

But if x and y are in terms of 'u' and 'v' or other variables, differentiate partially w.r.to the ~~ge~~ variable asked in the qn.

$x = 3t$ | $y = 4t^3$

$\frac{dx}{dt} = 3$ | $\frac{dy}{dt} = 4(3t^2) = 12t^2$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$u = \sin^{-1}(x-y)$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}}(1-$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-(x-y)^2}}(0-1)$$

$$\therefore \frac{du}{dt} = \frac{1}{\sqrt{1-(x-y)^2}}\cdot(3$$

$$- \frac{1}{\sqrt{1-(x-y)^2}}(12$$

$$\frac{du}{dt} = \frac{3}{\sqrt{1-(x-y)^2}} - \frac{12t^2}{\sqrt{1-(x-y)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(x-y)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(9t^2+16t^6-24t^4)}}$$