Maxima and Minima for functions of two variables.

Necessary conditions for a maximum or a minimum.

Given function $f(x, y)$

at the point (a, b)

Necessary condition is $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

at the point (a, b)

i.e., $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Sufficient Conditions:

If $f_x(a, b) = 0$; $f_y(a, b) = 0$ and

Let $f_{xx}(a, b) = A$; $f_{xy}(a, b) = B$ and

$f_{yy}(a, b) = C$.

i.e., $A = \frac{\partial^2 f}{\partial x^2}$; $B = \frac{\partial^2 f}{\partial x \partial y}$; $C = \frac{\partial^2 f}{\partial y^2}$

1) If $AC - B^2 > 0$ and $A > 0$	Minimum Value
2) If $AC - B^2 > 0$ and $A < 0$	Maximum Value
3) If $AC - B^2 < 0$	Neither maximum nor minimum (Saddle point) extremum.
4) If $AC - B^2 = 0$	Inconclusive

Critical Point or a Stationary point

A function $f(x,y)$ is said to be stationary at (a,b) (or) if (a,b) is called as the critical point or stationary point of the function $f(x,y)$, if $f_x(a,b)=0$ and

$$f_y(a,b)=0$$

i.e., $\frac{\partial f}{\partial x}=0$ at (a,b) and

$\frac{\partial f}{\partial y}=0$ at (a,b).

Example-1: Find the maximum or minimum values of $3x^2-y^2+x^3$.

Given: $f(x,y)=3x^2-y^2+x^3$

$\frac{\partial f}{\partial x}=f_x=6x+3x^2$ | $\frac{\partial f}{\partial y}=f_y=-2y$ | $\frac{\partial^2 f}{\partial x^2}=6+6x$

$\frac{\partial^2 f}{\partial y^2}=-2$ | $\frac{\partial^2 f}{\partial x \partial y}=0$.

Necessary condition: $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$

$6x+3x^2=0$; $-2y=0$

$3x(2+x)=0$ $\Rightarrow y=0$

$\Rightarrow x=0$; $x=-2$.

Hence the points at which maxima or minima exists are $(0,0)$ and $(-2,0)$.

At $(0,0)$

$A=\frac{\partial^2 f}{\partial x^2}=6$ | $B=\frac{\partial^2 f}{\partial x \partial y}=0$ | $C=\frac{\partial^2 f}{\partial y^2}=-2$.

$$AC - B^2 = 6(-2) - 0 = -12 < 0.$$

$\therefore$ The point is neither a maximum point nor a minimum point.

At $(-2, 0)$,

$$A = \frac{\partial^2 f}{\partial x^2} = 6 + 6(-2) = 6 - 12 = -6$$

$$B = \frac{\partial^2 f}{\partial x \partial y} = 0 \quad \Big| \quad C = \frac{\partial^2 f}{\partial y^2} = -2$$

$$AC - B^2 = (-6)(-2) - 0 = 12 > 0.$$

and $A = -6 < 0$.

$\therefore$ The point $(-2, 0)$ is a maximum point.

$\therefore$ Maximum Value of f is

$$3(-2)^2 - (0) + (-2)^3 = 12 - 8 = 4 \,//.$$

$f = 4$ is the maximum value.

(or)

Critical Point	A	B	C	$AC-B^2$	Conclusion
$(0,0)$	6	0	-2	-12	neither max nor min (saddle point)
$(-2,0)$	-6	0	-2	12	maximum point

$\therefore$ Maximum Value $= f(-2, 0) = \underline{\underline{4}}$

2. Find the maxima and minima of the function

$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

Soln:

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72$$

$$\frac{\partial f}{\partial y} = +6xy - 30y.$$

$\frac{\partial f}{\partial x} = 0$; $\frac{\partial f}{\partial y} = 0$ Necessary conditions

$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72 = 0$ — (1)

$6xy - 30y = 0$

$6y(x-5) = 0$

$6y = 0$ / $x - 5 = 0$

$y = 0$ $\quad x = 5$

put $y = 0$ in (1)

$3x^2 - 30x + 72 = 0$. $\quad 24 \to -6, -4$.

$x^2 - 10x + 24 = 0$

$(x-6)(x-4) = 0$

$x = 4, 6$.

pts are $(4,0)$ & $(6,0)$

put $x = 5$ in (1).

$3(25) + 3y^2 - 150 + 72 = 0$.

$75 + 3y^2 - 150 + 72 = 0$.

$3y^2 - 3 = 0$

$y^2 = 1$

$y = \pm 1$

pts are $(5,1)$ & $(5,-1)$.

∴ The critical points are

$(4,0)$; $(6,0)$; $(5,1)$; $(5,-1)$.

$$A = \frac{\partial^2 f}{\partial x^2} = 6x$$

$$B = \frac{\partial^2 f}{\partial x \partial y} = 6y$$

$$C = \frac{\partial^2 f}{\partial y^2} = 6x - 30$$

Critical point	A	B	C	$AC-B^2$	Conclusion
$(4,0)$	-6	0	-6	36	Maximum
$(6,0)$	6	0	6	36	Minimum
$(5,1)$	0	6	0	-36	Saddle point
$(5,-1)$	0	-6	0	-36	Saddle point.

Maximum at $(4,0)$; $f(4,0) = 112$

Minimum at $(6,0)$; $f(6,0) = 108$.

H.W: 1) Find Maxima and Minima for

$f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$ C.P $(0,0)(2,0)$ $(1,1)(1,-1)$

Max $(0,0)$ Min $(2,0)$

2) Find Maxima and Minima for

$f(x,y) = x^3 + y^3 - 12x - 3y + 20$. C.P $(2,1)(2,-$ $(-2,1),(-2,$

Max $(-2,-1)$ Min $(2,1)$.