Method of variation of parameter

If $\phi(x)$ or $Q(x) = \sec ax$ (or) $\tan ax$ then $P.I = Pf_1 + Qf_2$

where $P = -\int \frac{f_2 x}{W} dx$, $Q = \int \frac{f_1 x}{W} dx$ $W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$

Example 1: Solve by method of variation of parameter.

$(D^2+1)y = \sec x$

Sol

Aux eqn is $m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$

$C.F = e^{0x}(A\cos x + B\sin x)$

$f_1 = \cos x \qquad f_2 = \sin x$

$f_1' = -\sin x \qquad f_2' = \cos x$

$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$W = \cos^2 x + \sin^2 x = 1$

$P.I = Pf_1 + Qf_2$

$P = -\int \frac{f_2 x}{W} dx$

$= -\int \frac{\sin x \sec x}{1} dx$

$$= -\int \sin x \frac{1}{\cos x}\, dx$$

$$= -\int \tan x \, dx$$

$$= -\log \sec x$$

$$Q = \int \frac{f_1 x}{W}\, dx$$

$$= \int \frac{\cos x \sec x}{1}\, dx$$

$$= \int \cos x \frac{1}{\cos x}\, dx = \int dx$$

$$= x$$

$$P.I = P f_1 + Q f_2$$

$$= -\log(\sec x)(\cos x) + x \sin x$$

$$y = C.F + P.I$$

$$= A\cos x + B\sin x - \log(\sec x)\cos x + x\sin x$$

(2) Solve $(D^2+1)y = \operatorname{cosec} x$

Aux eqn $m^2+1 = 0$

$$m^2 = -1$$

$$m = \pm i$$

$$C.F = e^{0x}(A\cos x + B\sin x)$$

$f_1 = \cos x \qquad f_2 = \sin x$

$f_1' = -\sin x \qquad f_2' = \cos x$

$$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$W = \cos^2 x + \sin^2 x = 1$$

$P.I = P f_1 + Q f_2$

$$P = -\int \frac{f_2 x}{w} dx \quad , \quad Q = \int \frac{f_1 x}{w} dx$$

$$P = -\int \frac{\sin x \ \operatorname{cosec} x}{1} dx$$

$$= -\int \sin x \frac{1}{\sin x} dx$$

$$= -\int dx = -x$$

$$Q = \int \frac{\cos x \ \operatorname{cosec} x}{1} dx$$

$$= \int \cos x \frac{1}{\sin x} dx$$

$$= \int \cot x \, dx$$

$$= \log(\sin x)$$

$$P.I = -x \cos x + \log(\sin x) \sin x$$

$$y = A \cos x + B \sin x - x \cos x + \log(\sin x) \sin x$$

(3) Solve $(D^2+1)y = \tan x$

Aux eqn $m^2+1=0$

$m = \pm i$

$$C.F = A \cos x + B \sin x$$

$$\boxed{y = C.F + P.I}$$

$f_1 = \cos x$ $\quad$ $f_2 = \sin x$

$f_1' = -\sin x$ $\quad$ $f_2' = \cos x$

$$w = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$P.I = P f_1 + Q f_2$

$$P = -\int \frac{f_2 x}{w} dx$$

$$= -\int \frac{\sin x \tan x}{1} dx$$

$$= -\int \sin x \frac{\sin x}{\cos x} dx = -\int \frac{\sin^2 x}{\cos x} dx$$

$$= -\int \frac{1-\cos^2 x}{\cos x} dx = -\int \frac{1}{\cos x} dx + \int \cos x \, dx$$

$$P = -\int \sec x \, dx + \int \cos x \, dx$$

$$= -\log(\sec x + \tan x) + \sin x$$

$$Q = \int \frac{f_1 \, \phi x}{W} dx$$

$$= \int \frac{\cos x \, \tan x}{1} dx$$

$$= \int \cancel{\cos x} \frac{\sin x}{\cancel{\cos x}} dx$$

$$= -\cos x$$

$$P.I = P f_1 + Q f_2$$

$$= -\log(\sec x + \tan x)\cos x$$
$$- \cos x$$

$$y = C.F + P.I$$

$$= A\cos x + B\sin x - \log(\sec x + \tan x)$$
$$- \cos x$$

Cauchy's homogeneous linear equation

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots a_n y = \phi(x) \text{ (or) } Q(x)$$

Put $x = e^t$ (or) $t = \log x$, then if $\theta = \frac{d}{dt}$

$$x \frac{dy}{dx} = xDy = \theta y$$

$$x^2 \frac{d^2 y}{dx^2} = x^2 D^2 y = \theta(\theta - 1) y = (\theta^2 - \theta) y$$

$$x^3 \frac{d^3 y}{dx^3} = x^3 D^3 y = \theta(\theta - 1)(\theta - 2) y$$