JACOBIANS

If $u = f(x,y)$ & $v = g(x,y)$ be the two cts. functions of x & y then the functional determinant

$$|J| = \frac{\partial(u,v)}{\partial(x,y)} = \begin{array}{c} \\ u \\ v \end{array}\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$ is called

Jacobians of u and v with respect to x & y.

Three functions & three variables

$$|J| = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties:

1) If u, v are functions of x & y and x, y are functions of r & s then

$$\frac{\partial(u,v)}{\partial(r,s)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,s)}$$

2) If u & v are functions of x & y then

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$$

3) If u, v, w are functionally dependent functions which are depends on x, y & z then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

① If $u = \frac{y^2}{x}$, $v = \frac{x^2}{y}$ Find $\frac{\partial(u,v)}{\partial(x,y)}$.

Soln: $u = \frac{y^2}{x}$; $v = \frac{x^2}{y}$

$\frac{\partial u}{\partial x} = -\frac{y^2}{x^2}$; $\frac{\partial v}{\partial x} = \frac{2x}{y}$

$\frac{\partial u}{\partial y} = \frac{2y}{x}$; $\frac{\partial v}{\partial y} = -\frac{x^2}{y^2}$

$$|J| = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} -y^2/x^2 & 2y/x \\ 2x/y & -x^2/y^2 \end{vmatrix}$$

$$= -\frac{y^2}{x^2} \times -\frac{x^2}{y^2} - \frac{2y}{x} \times \frac{2x}{y}$$

$$= 1-4$$

$$= -3$$

② If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

$u = \frac{yz}{x}$; $v = \frac{zx}{y}$; $w = \frac{xy}{z}$

$\frac{\partial u}{\partial x} = -\frac{yz}{x^2}$; $\frac{\partial v}{\partial x} = \frac{z}{y}$; $\frac{\partial w}{\partial x} = \frac{y}{z}$

$\frac{\partial u}{\partial y} = \frac{z}{x}$; $\frac{\partial v}{\partial y} = -\frac{zx}{y^2}$; $\frac{\partial w}{\partial y} = \frac{x}{z}$

$\frac{\partial u}{\partial z} = \frac{y}{x}$; $\frac{\partial v}{\partial z} = \frac{x}{y}$; $\frac{\partial w}{\partial z} = -\frac{xy}{z^2}$

$$|J| = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= -\frac{yz}{x^2}\left[-\frac{zx}{y^2} \times -\frac{xy}{z^2} - \frac{x}{y} \times \frac{x}{z}\right] - \frac{z}{x}\left[-\frac{xy}{z^2} \times \frac{z}{y} - \frac{x}{y} \cdot \frac{y}{z}\right] +$$

$$\frac{y}{x}\left[\frac{z}{y} \times \frac{x}{z} - \frac{y}{z} \times -\frac{zx}{y^2}\right]$$

$= 4.$

(3) If $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(x,y)}{\partial(r,\theta)}$

Soln:

$x = r\cos\theta$; $y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta$; $\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta$; $\frac{\partial y}{\partial \theta} = r\cos\theta$

$$|J| = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

(4) If $x = uv$, $y = \frac{u+v}{u-v}$ find $\frac{\partial(u,v)}{\partial(x,y)}$

Soln:

WKT $\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = 1$

$$J \cdot J' = 1$$

$$|J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$x = uv \quad ; \quad y = \frac{u+v}{u-v}$

$\frac{\partial x}{\partial u} = v$

$\frac{\partial x}{\partial v} = u$

$\frac{\partial y}{\partial u} = \frac{(u-v)-(u+v)}{(u-v)^2} = \frac{-2v}{(u-v)^2}$

$\frac{\partial y}{\partial v} = \frac{(u-v)-(u+v)(-1)}{(u-v)^2} = \frac{2u}{(u-v)^2}$

$\therefore |J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v & u \\ \frac{-2v}{(u-v)^2} & \frac{2u}{(u-v)^2} \end{vmatrix}$

$= \frac{2uv}{(u-v)^2} + \frac{2uv}{(u-v)^2} = \frac{4uv}{(u-v)^2}$

$\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = 1$

$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}} = \frac{1}{\frac{4uv}{(u-v)^2}}$

$= \frac{(u-v)^2}{4uv}$

2) If $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(r,\theta)}{\partial(x,y)}$

If $u = 2xy$, $v = x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$.

find $\dfrac{\partial(u,v)}{\partial(r,\theta)}$

Soln:
$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

Gn: $u = 2xy$ $\qquad v = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2y \qquad \frac{\partial v}{\partial x} = 2x$$
$$\frac{\partial u}{\partial y} = 2x \qquad \frac{\partial v}{\partial y} = -2y$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} = -4y^2 - 4x^2$$

Gn: $x = r\cos\theta$ $\qquad y = r\sin\theta$.

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$
$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$\therefore \frac{\partial(u,v)}{\partial(r,\theta)} = (-4y^2-4x^2) \times r.$

$= -4(x^2+y^2) \times r$

$= -4r^2 \times r$

$= -4r^3.$

$x = r\cos\theta, y = r\sin\theta$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2.$

(10) S.T. the functions $u = \frac{x}{y}$ & $v = \frac{x+y}{x-y}$ are functionally dependent and find the relationship btwn. them.

Soln: $u = \frac{x}{y}$; $v = \frac{x+y}{x-y}$

$\frac{\partial u}{\partial x} = \frac{1}{y}$ $\qquad \frac{\partial v}{\partial x} = \frac{x-y-(x+y)(+1)}{(x-y)^2} = \frac{-2y}{(x-y)^2}$

$\frac{\partial u}{\partial y} = -\frac{x}{y^2}$ $\qquad \frac{\partial v}{\partial y} = \frac{x-y-(x+y)(-1)}{(x-y)^2} = \frac{2x}{(x-y)^2}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1}{y} & -\frac{x}{y^2} \\ \frac{-2y}{(x-y)^2} & \frac{2x}{(x-y)^2} \end{vmatrix}$$

$= 0$

$\therefore$ the gn. functions are functionally dependent

Gn: $u = x/y$, $v = \frac{x+y}{x-y}$.

$$v = \frac{\left[\frac{x}{y}+1\right]y}{\left[\frac{x}{y}-1\right]y} = \frac{u+1}{u-1}$$

$$\Rightarrow v = \frac{u+1}{u-1}$$

(11) S.T. The functions $u = 2x - y + 3z$, $v = 2x - y - z$, $w = 2x - y + z$ are functionally dependent. Find relationship between them.

Ans: Functionally dependent.

Relationship: $u + v = 2w$.

(12) If $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$ & $w = x + y + z$ determine whether a function relation btwn x, y, z are dependent & find relationship btwn. them.

Ans: $2u + v = w^2$ (relationship)