(6)

Envelope:

The envelope of the family of curves is a curve which touches each membrane of a family.

Procedure to find the envelope of the family of curves:

Method 1:

1) Differentiate $f(x,y,c)=0$ partially w.r.t. parameter 'c'.
2) Eliminate c from $f(x,y,c)=0$ & $\frac{\partial}{\partial c} f(x,y,c)=0$
3) The eqn. after the elimination of c is the envelope of the family.

Method 2:

If the family of curves is expressed as the quadratic form say $Aa^2+Ba+C=0$ where a is the parameter Then the envelope of family of curve is $B^2-4AC=0$.

1) Find the envelope of $y=mx+am^2$.

$am^2+mx-y=0$

Here $A=a$; $B=x$, $C=-y$.

WKT $B^2-4AC=0$

$\Rightarrow x^2+4ay=0$

2) Find the envelope of $y = mx + a\sqrt{1+m^2}$ where m is the parameter.

$$y = mx + a\sqrt{1+m^2}$$

$$y - mx = a\sqrt{1+m^2}$$

$$\frac{y-mx}{a} = \sqrt{1+m^2}$$

Squaring OBS we get

$$\left(\frac{y-mx}{a}\right)^2 = 1+m^2$$

$$(y-mx)^2 = a^2(1+m^2)$$

$$y^2 + m^2x^2 - 2mxy = a^2 + a^2m^2$$

$$\Rightarrow m^2(x^2 - a^2) + m(-2xy) + (y^2 - a^2) = 0$$

$$\Rightarrow m^2[x^2 - a^2] - m[2xy] + [y^2 - a^2] = 0$$

Here $A = x^2 - a^2$; $B = -2xy$; $C = y^2 - a^2$

$$B^2 - 4AC = 0$$

$$\Rightarrow 4x^2y^2 - 4(x^2 - a^2)(y^2 - a^2) = 0$$

$$\Rightarrow 4x^2y^2 - 4[x^2y^2 - a^2x^2 - a^2y^2 + a^4] = 0$$

$$\Rightarrow 4x^2y^2 - 4x^2y^2 + 4a^2x^2 + 4a^2y^2 - 4a^4 = 0$$

$$\Rightarrow 4a^2x^2 + 4a^2y^2 - 4a^4 = 0$$

$$\Rightarrow x^2 + y^2 = a^2$$

(3) Find the envelope of $x\cos\theta + y\sin\theta = a$, where θ is parameter

$$x\cos\theta + y\sin\theta = a \quad \text{--- (1)}$$

partially d.w.r.t. θ.

$$-x\sin\theta + y\cos\theta = 0 \quad \text{--- (2)}$$

Taking sequare in (1) & (2).

$$(x\cos\theta + y\sin\theta)^2 = a^2 \quad \text{--- (3)}$$

$$(-x\sin\theta + y\cos\theta)^2 = 0 \quad \text{--- (4)}$$

$$(3)+(4) \Rightarrow x^2\cos^2\theta + y^2\sin^2\theta + 2xy\cos\theta\sin\theta$$
$$+ x^2\sin^2\theta + y^2\cos^2\theta - 2xy\cos\theta\sin\theta = a^2$$

$$\Rightarrow x^2 + y^2 = a^2$$

(4) Find the envelope of $x\sec\theta - y\tan\theta = a$ where θ is parameter.

$$x\sec\theta - y\tan\theta = a \quad \text{--- (1)}$$

partially d.w.r.t. θ.

$$x\sec\theta\tan\theta - y\sec^2\theta = 0 \quad \text{--- (2*)}$$

$$(x\tan\theta - y\sec\theta)\sec\theta = 0$$

$$\Rightarrow x\tan\theta - y\sec\theta = 0 \quad \text{--- (2)}.$$

Taking square in ① & ②

$$(x \sec\theta - y\tan\theta)^2 = a^2 \quad \text{——} ③$$

$$(x\tan\theta - y\sec\theta)^2 = 0 \quad \text{——} ④$$

③ - ④

$$\Rightarrow x^2\sec^2\theta + y^2\tan^2\theta - 2xy\sec\theta\tan\theta$$
$$- [x^2\tan^2\theta + y^2\sec^2\theta - 2xy\sec\theta\tan\theta] = a^2$$

$$\Rightarrow x^2\sec^2\theta + y^2\tan^2\theta - 2xy\sec\theta\tan\theta$$
$$- x^2\tan^2\theta - y^2\sec^2\theta + 2xy\sec\theta\tan\theta = a^2$$

$$\Rightarrow x^2 + y^2 = a^2$$

⑤ $\frac{x}{a}\cos t + \frac{y}{b}\sin t = 1$, t is a parameter.

Partially w.r.t. 't' in $\frac{x}{a}\cos t + \frac{y}{b}\sin t = 1$ —— ①

$$-\frac{x}{a}\sin t + \frac{y}{b}\cos t = 0 \quad \text{——} ②$$

Taking square in ① & ②

$$\left(\frac{x}{a}\cos t + \frac{y}{b}\sin t\right)^2 = 1 \quad \text{——} ③$$

$$\left(-\frac{x}{a}\sin t + \frac{y}{b}\cos t\right)^2 = 0 \quad \text{——} ④$$

③ + ④ $\Rightarrow \frac{x^2}{a^2}\cos^2 t + \frac{y^2}{b^2}\sin^2 t + 2\frac{x}{a}\cdot\frac{y}{b}\cos t\sin t$

$$+ \frac{x^2}{a^2}\sin^2 t + \frac{y^2}{b^2}\cos^2 t - 2\frac{x}{a}\cdot\frac{y}{b}\cos t\sin t = 1$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(6) $y = mx - 2am - am^3$, m is parameter.

$y = mx - 2am - am^3$ —— (1).

partially d.w.r.t. 'm'

$0 = x - 2a - 3am^2$

$\Rightarrow x - 2a = 3am^2$ —— (2)

Consider (1), $y = m[x - 2a - am^2]$

$= m[(x-2a) - am^2]$

From (2), $m^2 = \dfrac{x-2a}{3a}$

$$\Rightarrow y = m\left[(x-2a) - \cancel{a}\left[\frac{x-2a}{3\cancel{a}}\right]\right]$$

$$y = m\left[\frac{2}{3}[x-2a]\right]$$

$$\Rightarrow m = \frac{3y}{2[x-2a]}$$

Sub m in (2), $x - 2a - 3am^2 = 0$

$$(x-2a) - 3a\left[\frac{3y}{2[x-2a]}\right]^2 = 0$$

$$\Rightarrow x - 2a - 3a \cdot \frac{9y^2}{4[x-2a]^2} = 0$$

$$\Rightarrow 4[x-2a]^3 - 27ay^2 = 0.$$

(7) $\frac{x^2}{\alpha} + \frac{y^2}{1-\alpha} = 1$. , α is parameter.

$(1-\alpha)x^2 + \alpha y^2 = \alpha(1-\alpha)$.

$\Rightarrow (1-\alpha)x^2 + \alpha y^2 - \alpha + \alpha^2 = 0$

$\Rightarrow \alpha^2 + (-x^2+y^2-1)\alpha + x^2 = 0$

$A = 1$; $B = (-x^2+y^2-1)$; $C = x^2$.

$B^2 - 4AC = 0$

$\Rightarrow [-x^2+y^2-1]^2 - 4x^2 = 0$

$\Rightarrow x^4 + y^4 + 1 - 2x^2y^2 - 2y^2 + 2x^2 - 4x^2 = 0$

$\Rightarrow x^4 + y^4 + 1 - 2x^2y^2 - 2y^2 - 2x^2 = 0$

(8) $y = mx + \sqrt{a^2m^2+b^2}$, m is parameter.

$y - mx = \sqrt{a^2m^2+b^2}$

Taking square OBS.

$(y-mx)^2 = a^2m^2 + b^2$.

$\Rightarrow y^2 + m^2x^2 - 2mxy - a^2m^2 - b^2 = 0$

$\Rightarrow m^2(x^2-a^2) + m(-2xy) + y^2 - b^2 = 0$

$A = (x^2 - a^2)$; $B = -2xy$; $C = y^2 - b^2$.

$B^2 - 4AC = 0$

$\Rightarrow 4x^2y^2 - 4(x^2-a^2)(y^2-b^2) = 0$

$\Rightarrow 4x^2y^2 - 4[x^2y^2 - x^2b^2 - a^2y^2 + a^2b^2] = 0$

$\Rightarrow 4x^2b^2 + 4a^2y^2 - 4a^2b^2 = 0$

$\Rightarrow x^2b^2 + a^2y^2 = a^2b^2$.

$\Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

9. $y = mx + \dfrac{a}{m}$, m is parameter.

$\Rightarrow m^2x - my + a = 0$

$A = x$; $B = -y$; $C = a$.

$B^2 - 4AC = 0$

$y^2 - 4ax = 0$.