Total probability theorem :

If $B_1, B_2, \dots B_n$ are mutually exclusive and exhaustive set of events of a Sample space S and A be any event associated with the events $B_1, B_2, \dots B_n$. Then

$$P(A) = \sum_{i=1}^{n} P(B_i)\, P(A/B_i)$$

Baye's theorem :

If $B_1, B_2, \dots B_n$ are mutually exclusive and exhaustive events of a Sample Space S and A be any event associated with the events $B_1, B_2, \dots B_n$. Then

$$P(B_i | A) = \frac{P(B_i)\, P(A/B_i)}{\sum_{i=1}^{n} P(B_i)\, P(A/B_i)}$$

Problems :

(1) The content of bags I, II and III are as follows :

(a) 1 white, 2 black, 3 red balls

(b) 2 white, 1 black, 1 red balls

(c) 4 white, 5 black, 3 red balls

One bag is chosen at random and two balls are drawn. They happen to be white and red balls. What is the probability that they come from bag I, II and III ?

Solution :

There are 3 bags. The probability of choosing one bag is $1/3$.

(10)

Let B_1 be the event of choosing bag I.

Let B_2 be the event of choosing bag II.

Let B_3 be the event of choosing bag III.

i.e., $P(B_1) = 1/3$, $P(B_2) = 1/3$, $P(B_3) = 1/3$.

A be the event of getting 1 white ball and 1 red ball. Then,

$$P(A/B_1) = \frac{1C_1 \times 3C_1}{6C_2} = 1/5$$

$$P(A|B_2) = \frac{2C_1 \times 1C_1}{4C_2} = 1/3$$

$$P(A|B_3) = \frac{4C_1 \times 3C_1}{12C_2} = 2/11$$

By Baye's theorem,

$$P(B_i|A) = \frac{P(B_i)\,P(A|B_i)}{\sum_{i=1}^{3} P(B_i)\,P(A|B_i)}$$

$$P(B_1|A) = \frac{\frac{1}{3} \times \frac{1}{5}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{11}} = 0.277 = 2$$

$$P(B_2|A) = \frac{\frac{1}{3} \times \frac{1}{3}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{11}} = 0.4661 = 4$$

$$P(B_3/A) = \frac{\frac{1}{3} \times \frac{2}{11}}{\frac{1}{3} \cdot \frac{1}{5} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{11}} = 0.2542 = 2$$

(2) In a bolt factory machines A, B and C manufacture respectively 25%, 35%, 40% of the total of their output 5%, 4%, 2% are defective bolts. A bolt is drawn at random from the product and is found to defective. What are probabilities that it was manufactured by machine A, B, C?

Solution:

Let B_1 be the event that a bolt is manufactured by machine A, B_2 be the event that a bolt is manufactured by machine B, B_3 be the event that a bolt is manufactured by machine C.

Let A be the event that a bolt is defective.

$P(B_1) = 0.25$, $P(A|B_1) = 0.05$

$P(B_2) = 0.35$, $P(A|B_2) = 0.04$

$P(B_3) = 0.40$, $P(A|B_3) = 0.02$

By Baye's theorem,

$$P(B_i|A) = \frac{P(B_i)\,P(A|B_i)}{\sum_{i=1}^{n} P(B_i)\,P(A|B_i)}$$

P (Bolt was manufactured by machine A)

$$P(B_1|A) = \frac{P(B_1)\,P(A|B_1)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)}$$

$$= \frac{0.25 \times 0.05}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.4 \times 0.02}$$

$P(B_1|A) = 0.3623$

$$P(B_2|A) = \frac{P(B_2) \cdot P(A|B_2)}{P(B_1)P(A|B_1) + P(B_2)\,P(A|B_2) + P(B_3)\,P(A|}$$

$$= \frac{0.35 \times 0.04}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.4 \times 0.02}$$

$$\boxed{P(B_2|A) = 0.4057}$$

$$P(B_3|A) = \frac{P(B_3) \cdot P(A|B_3)}{P(B_1)\,P(A|B_1) + P(B_2)\,P(A|B_2) + P(B_3)\,P(A|}$$

$$= \frac{0.40 \times 0.02}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.4 \times 0.02}$$

$$\boxed{P(B_3|A) = 0.2318}$$

(3) The chances of 3 candidates A, B and C becoming t manager of a company are in the ratio 3 : 5 : 4. The probabilities that a special bonus scheme will be introduc by them if selected are 0.6, 0.4 and 0.5 respectively. I the bonus scheme is introduced, what is the probability that B has become the manager?

Solution:

Let B_1, B_2 & B_3 be the event of selecting A, B and C as manager of a company.

$$P(B_1) = \frac{3}{12} = \frac{1}{4}$$

$$P(B_2) = \frac{5}{12} = \frac{5}{12}$$

$$P(B_3) = \frac{4}{12} = \frac{1}{3}$$

Let A be the event of introducing special bonus scheme.

$P(A/B_1) = 0.6$

$P(A/B_2) = 0.4$

$P(A/B_3) = 0.5$

By Baye's theorem,

$$P(B_2/A) = \frac{P(A/B_2).P(B_2)}{P(A/B_1)P(B_1) + P(A/B_2).P(B_2) + P(A/B_3)P(B_3)}$$

$$= \frac{(0.4)(5/12)}{(0.6)(1/4) + (0.4)(5/12) + (0.5)(1/3)}$$

$$= \frac{0.1667}{0.4833} = 0.3449 = 34.49\%.$$

(4) A company has two plants. Plant I manufactures 25% of the items. Plant II manufactures 75% of the items. 3% and 5% of the items manufactured by plant I and II respectively are known to be defective. What is the chance that it was generated by plant II.

Solution:

Let B_1 & B_2 be the event manufactured by plant I and II respectively.

$P(B_1) = 25\% = 0.25$

$P(B_2) = 75\% = 0.75$

Let A be the event that the item is defective.

$P(A/B_1) = 3\% = 0.03$

$P(A/B_2) = 5\% = 0.05$