BEAM ELEMENT

A beam is a long, slender structural member generally subjected to transverse loading that produces significant bending effects as opposed to twisting or axial effects. An elemental length of a long beam subjected to arbitrary loading is considered for analysis. For this elemental beam length L, we assign two points of interest, i.e., the ends of the beam, which become the nodes of the beam element. The bending deformation is measured as a transverse (vertical) displacement and a rotation (slope). Hence, for each node, we have a vertical displacement and a rotation (slope) – two degrees of freedom at each node. For a single 2-noded beam element, we have a total of 4 degrees of freedom. The associated "forces" are shear force and bending moment at each node.

Nodal "displacements"

Nodal "forces"

1st degree of freedom	vertical displacement at node *i*	1	v_i or v_1		shear force at node *i*	F_i or F_1	1
2nd degree of freedom	slope or rotation at node *i*	2	θ_i or θ_1	corres- pond- ing to	bending moment at node *i*	M_i or M_1	2
3rd degree of freedom	vertical displacement at node *j*	3	v_j or v_2		shear force at node *i*	F_j or F_2	3
4th degree of freedom	slope or rotation at node *j*	4	θ_j or θ_2		bending moment at node *j*	M_j or M_2	4

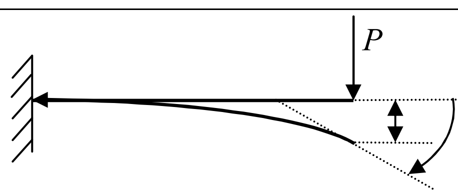
The stiffness term k_{ij} indicates the force (or moment) required at *i* to produce a unit deflection (or rotation) at *j*, while all other degrees of freedom are kept zero.

Sign conventions followed

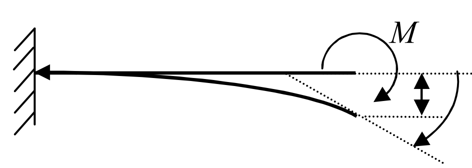
Upward forces are positive and upward displacements are positive.

Counter-clockwise moments are positive and counter-clockwise rotations are positive.

Formulae required – cantilever beam subjected to concentrated load and moment.



$$\theta = \frac{PL^2}{2EI} \qquad \delta = \frac{PL^3}{3EI}$$



$$\theta = \frac{ML}{EI} \qquad \delta = \frac{ML^2}{2EI}$$

DERIVATION OF STIFFNESS MATRIX OF BEAM ELEMENT

Derivation of first column of stiffness matrix: $v_1 = 1,\ \theta_1 = v_2 = \theta_2 = 0$, i.e., allow the first degree of freedom to occur and arrest all other DoF. (The deformed configuration is shown in Figure 2).

Initially you have a horizontal beam element. Since $v_2 = \theta_2 = 0$, we can fix node *j*. To produce an upward deflection at node *i* (i.e., allowing first degree of freedom to occur), apply an upward force k_{11} (first suffix indicates the force or moment DoF and the second suffix indicates the displacement or rotational DoF). $v_1 = \dfrac{k_{11}L^3}{3EI}$ upwards. Refer table for displacement DoF number and force DoF number. Now the beam configuration is given by Figure 1. We can observe from the figure that the slope at node *i* is not zero. To make the slope at *i* equal to zero, we need to apply a counter-clockwise moment k_{21}. Refer Figure 2. But this moment k_{21} will produce a downward deflection $\dfrac{k_{21}L^2}{2EI}$ at node *i*. Refer Figure 3. In order to have a resultant unit upward displacement at node *i*, upward displacement produced by force k_{11} must be greater than the downward displacement produced by the moment k_{21}. i.e., $\dfrac{k_{11}L^3}{3EI} - \dfrac{k_{21}L^2}{2EI} = 1$ …..(1). At the same time, the negative slope produced at node *i* by the force k_{11} must be cancelled by the positive slope produced by the moment k_{21}. i.e., $\dfrac{k_{11}L^2}{2EI} = \dfrac{k_{21}L}{EI}$ ….(2). Solving these two equations, k_{11} and k_{21} are found. The fixed end reaction force and the reaction moment are assumed to be acting upwards and counterclockwise, respectively. Now use force equilibrium equation to find fixed end reaction force k_{31} …….$\left(\sum F_y = 0 \quad \Rightarrow \quad k_{11} + k_{31} = 0\right)$ and moment equilibrium equation about node *i* to find fixed end reaction moment k_{41} ….$\left(\sum M_i = 0 \quad \Rightarrow \quad k_{21} + k_{31}L + k_{41} = 0\right)$.

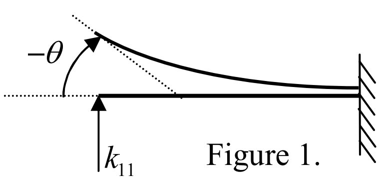


Figure 1.

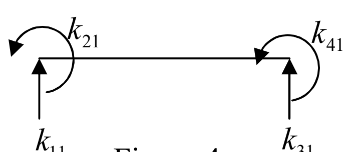


Figure 4.

$$\begin{bmatrix} k_{11} \\ k_{21} \\ k_{31} \\ k_{41} \end{bmatrix} = \begin{bmatrix} \dfrac{12EI}{L^3} \\ \dfrac{6EI}{L^2} \\ -\dfrac{12EI}{L^3} \\ \dfrac{6EI}{L^2} \end{bmatrix}$$

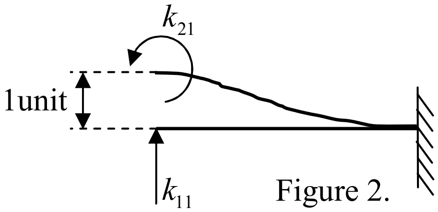


Figure 2.

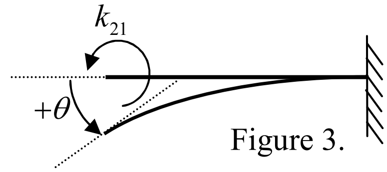


Figure 3.

Derivation of second column of stiffness matrix: $v_1 = 0,\ \theta_1 = 1, v_2 = \theta_2 = 0$, i.e., allow the second degree of freedom to occur and arrest all other DoF. (The deformed configuration is shown in Figure 2).

Initially you have a horizontal beam element. Since $v_2 = \theta_2 = 0$, we can fix node *j*. To produce a counterclockwise (positive) rotation or slope at node *i* (i.e., allowing second degree of freedom to occur), apply a counterclockwise moment $k_{22}.\theta_1 = \frac{k_{22}L}{EI}$. Refer Figure 1. This moment k_{22} will produce a downward deflection $\frac{k_{22}L^2}{2EI}$. This downward deflection should be canceled by applying an upward force k_{12} at node *i*. The upward deflection produced by k_{12} is $\frac{k_{12}L^3}{3EI}$. Refer Figure 2. Equating these two deflections $\frac{k_{22}L^2}{2EI} = \frac{k_{12}L^3}{3EI}$ …(1) But this upward force k_{12} will also produce a negative slope at node *i* which is $\frac{k_{12}L^2}{2EI}$. Refer Figure 3. Hence the rotation produced by k_{22} should be greater than that produced by k_{12} so that the resultant rotation is 1 radians. $\frac{k_{22}L}{EI} - \frac{k_{12}L^2}{2EI} = 1$….(2). Solving these two equations, k_{12} and k_{22} are found. The fixed end reaction force and the reaction moment are assumed to be acting upwards and counterclockwise, respectively. Now use force equilibrium equation to find fixed end reaction force k_{32} …$\left(\sum F_y = 0 \ \Rightarrow \ k_{12} + k_{32} = 0\right)$ and moment equilibrium equation about node *i* to find fixed end reaction moment k_{42}….$\left(\sum M_i = 0 \ \Rightarrow \ k_{22} + k_{32}L + k_{42} = 0\right)$.

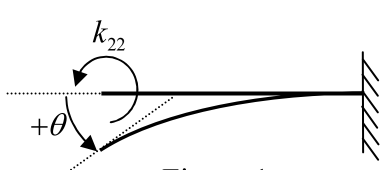


Figure 1.

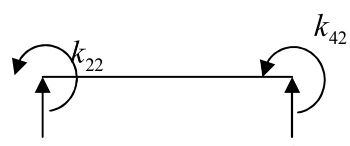


Figure 4.

$$\begin{bmatrix} k_{12} \\ k_{22} \\ k_{32} \\ k_{42} \end{bmatrix} = \begin{bmatrix} \frac{6EI}{L^2} \\ \frac{4EI}{L} \\ -\frac{6EI}{L^2} \\ \frac{2EI}{L} \end{bmatrix}$$

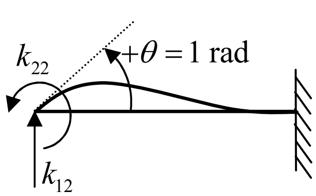


Figure 2.

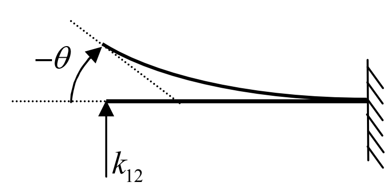


Figure 3.

Derivation of third column of stiffness matrix: $v_1 = 0,\ \theta_1 = 0, v_2 = 1, \theta_2 = 0$, i.e., allow the third degree of freedom to occur and arrest all other DoF. (The deformed configuration is shown in Figure 2).

Initially you have a horizontal beam element. Since $v_1 = \theta_1 = 0$, we can fix node *i*. To produce an upward deflection at node *j* (i.e., allowing third degree of freedom to occur), apply an upward force $k_{33} \cdot v_2 = \frac{k_{33}L^3}{3EI}$ upwards. Now the beam configuration is given by Figure 1. We can observe from the figure that the slope at node *j* is not zero. To make the slope at *j* equal to zero, we need to apply a clockwise moment k_{43}. Refer Figure 2. But this moment k_{43} will produce a downward deflection $\frac{k_{43}L^2}{2EI}$ at node *j*. Refer Figure 3. In order to have a resultant unit upward displacement at node *j*, upward displacement produced by force k_{33} must be greater than the downward displacement produced by the moment k_{43}. i.e., $\frac{k_{33}L^3}{3EI} - \frac{k_{43}L^2}{2EI} = 1$ …..(1). At the same time, the positive slope produced at node *j* by the force k_{33} must be cancelled by the negative slope produced by the moment k_{43}. i.e., $\frac{k_{33}L^2}{2EI} = \frac{k_{43}L}{EI}$ ….(2). Solving these two equations, k_{33} and k_{43} are found. The fixed end reaction force and the reaction moment are assumed to be acting upwards and counterclockwise, respectively. Now use force equilibrium equation to find fixed end reaction force k_{13} $\ldots\left(\sum F_y = 0 \ \Rightarrow\ k_{13} + k_{33} = 0\right)$ and moment equilibrium equation about node *i* to find fixed end reaction moment k_{23} ….$\left(\sum M_i = 0 \ \Rightarrow\ k_{23} + k_{33}L + k_{43} = 0\right)$.

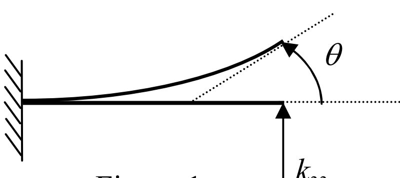


Figure 1.

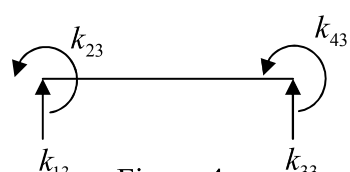


Figure 4.

$$\begin{bmatrix} k_{13} \\ k_{23} \\ k_{33} \\ k_{43} \end{bmatrix} = \begin{bmatrix} -\frac{12EI}{L^3} \\ -\frac{6EI}{L^2} \\ \frac{12EI}{L^3} \\ -\frac{6EI}{L^2} \end{bmatrix}$$

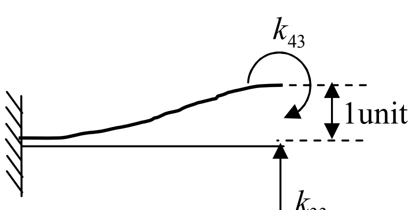


Figure 2.

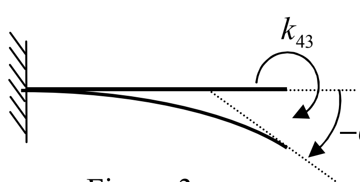


Figure 3.

Derivation of fourth column of stiffness matrix: $v_1 = \theta_1 = 0, v_2 = 0, \theta_2 = 1$, i.e., allow the fourth degree of freedom to occur and arrest all other DoF. (The deformed configuration is shown in Figure 2).

Initially you have a horizontal beam element. Since $v_1 = \theta_1 = 0$, we can fix node *i*. To produce a counterclockwise (positive) rotation or slope at node *j* (i.e., allowing fourth degree of freedom to occur), apply a counterclockwise moment $k_{44}.\theta_2 = \frac{k_{44}L}{EI}$. Refer Figure 1. This moment k_{44} will produce a upward deflection $\frac{k_{44}L^2}{2EI}$. This upward deflection should be canceled by applying a downward force k_{34} at node *j*. The downward deflection produced by k_{34} is $\frac{k_{34}L^3}{3EI}$. Refer Figure 2. Equating these two deflections $\frac{k_{44}L^2}{2EI} = \frac{k_{34}L^3}{3EI}$ …(1) But this downward force k_{34} will also produce a negative slope at node *j* which is $\frac{k_{34}L^2}{2EI}$. Hence the rotation produced by k_{44} should be greater than that produced by k_{34} so that the resultant rotation is 1 radians. $\frac{k_{44}L}{EI} - \frac{k_{34}L^2}{2EI} = 1$ ….(2) Refer Figure 3. Solving these two equations, k_{34} and k_{44} are found. The fixed end reaction force and the reaction moment are assumed to be acting upwards and counterclockwise, respectively. Now use force equilibrium equation to find fixed end reaction force k_{14} …$\left(\sum F_y = 0 \;\; \Rightarrow \;\; k_{14} + k_{34} = 0\right)$ and moment equilibrium equation about node *i* to find fixed end reaction moment k_{24} ….$\left(\sum M_i = 0 \;\; \Rightarrow \;\; k_{24} + k_{34}L + k_{44} = 0\right)$.

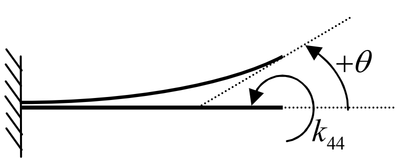


Figure 1.

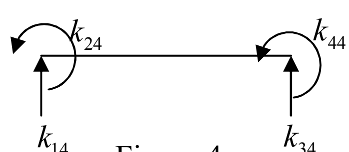


Figure 4.

$$\begin{bmatrix} k_{14} \\ k_{24} \\ k_{34} \\ k_{44} \end{bmatrix} = \begin{bmatrix} \frac{6EI}{L^2} \\ \frac{2EI}{L} \\ -\frac{6EI}{L^2} \\ \frac{4EI}{L} \end{bmatrix}$$

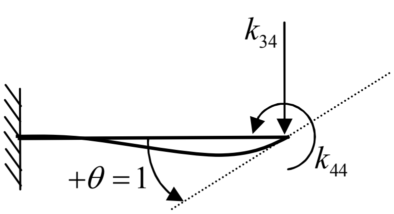


Figure 2.

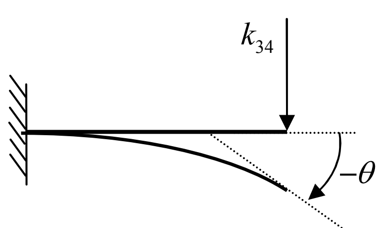


Figure 3.