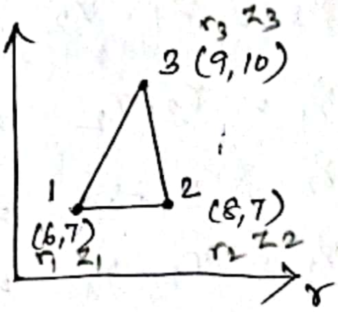
Calculate the element stiffness matrix and the thermal force vector for the axisymmetric element shown below. The element experiences a $15^\circ c$ increase in temp

Take,

$\alpha = 10 \times 10^{-6} /^\circ c$

$E = 2 \times 10^5 \ N/mm^2$

$\nu = 0.25$



To Find:

(i) Element stiffness matrix [K]

(ii) Thermal force vector $\{F\}_t$

Solution:-

$$[K] = 2\pi A \bar{r} \underset{6\times4}{[B]^T} \underset{4\times4}{[D]} [B]$$

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$\alpha_1 = r_2 z_3 - r_3 z_2 = (8 \times 10) - (9 \times 7) = 17 \ mm^2$$

$$\alpha_2 = r_3 z_1 - r_1 z_3 = (9 \times 7) - (6 \times 10) = 3 \ mm^2$$

$$\alpha_3 = r_1 z_2 - r_2 z_1 = (6 \times 7) - (8 \times 7) = -14 \ mm^2$$

$z_1 = 7;\ z_2 = 7;\ z_3 = 10$
$r_1 = 6;\ r_2 = 8;\ r_3 = 9$

$\beta_1 = z_2 - z_3 = -3\ \text{mm}$

$\beta_2 = z_3 - z_1 = 3\ \text{mm}$

$\beta_3 = z_1 - z_2 = 0$

$\gamma_1 = r_3 - r_2 = 1\ \text{mm}$

$\gamma_2 = r_1 - r_3 = -3\ \text{mm}$

$\gamma_3 = r_2 - r_1 = 2\ \text{mm}$

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{6+8+9}{3} = 23/3 \qquad ; \qquad z = \frac{z_1 + z_2 + z_3}{3} = \frac{7+7+10}{3} = 24/3$$

$r = 7.67\ \text{mm}$ $\qquad$ $z = 8\ \text{mm}.$

$$\frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} = \frac{17}{7.67} + (-3) + \frac{1(8)}{7.67} = 0.259\ \text{mm}$$

$$\frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} = \frac{3}{7.67} + (3) + \frac{-3(8)}{7.67} = 0.262\ \text{mm}$$

$$\frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} = \frac{-14}{7.67} + (0) + \frac{2(8)}{7.67} = 0.260\ \text{mm}.$$

$$\text{Area} = \frac{1}{2} \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 1 & r_1\,6 & z_1\,7 \\ 1 & r_2\,8 & z_2\,7 \\ 1 & r_3\,9 & z_3\,10 \end{vmatrix} \end{array}$$

$$= \frac{1}{2}\left[1(80-63) - 6(10-7) + 7(9-8)\right]$$

$$= \frac{1}{2}[6]$$

$$\boxed{A = 3\ \text{mm}^2}$$

$$B = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$= \frac{1}{2(3)}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0.259 & 0 & 0.262 & 0 & 0.260 & 0 \\ 0 & 1 & 0 & -3 & 0 & 2 \\ 1 & -3 & -3 & 3 & 2 & 0 \end{bmatrix}$$

$$B = 0.167\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0.259 & 0 & 0.262 & 0 & 0.260 & 0 \\ 0 & 1 & 0 & -3 & 0 & 2 \\ 1 & -3 & -3 & 3 & 2 & 0 \end{bmatrix}_{4\times 6}$$

$$B^T = 0.167\begin{bmatrix} -3 & 0.259 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 3 & 0.262 & 0 & -3 \\ 0 & 0 & -3 & 3 \\ 0 & 0.260 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}_{6\times 4}$$

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Stress strain relationship matrix [D]

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$$= \frac{2\times10^5}{(1+0.25)(1-\cancel{0.25}\,0.5)}\begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$[D] = 8\times10^4 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$[B]^T [D] =$$

$$= 13.36 \times 10^3 \begin{bmatrix} -3 & 0.259 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 3 & 0.262 & 0 & -3 \\ 0 & 0 & -3 & 3 \\ 0 & 0.260 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}_{6\times4} \times \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4}$$

$6\times4 \times 4\times4$

$$= 13.36\times10^3 \begin{bmatrix} (-9+0.259+0+0) & (-3+0.777+0+0) & (-3+0.259+0+0) & (0+0+0+1) \\ (0+0+1+0) & (0+0+1+0) & (0+0+3+0) & (0+0+0-3) \\ (9+0.262+0+0) & (3+0.786+0+0) & (3+0.262+0+0) & (0+0+0-3) \\ (0+0-3+0) & (0+0-3+0) & (0+0-9+0) & (0+0+0+3) \\ (0+0.260+0+0) & (0+0.78+0+0) & (0+0.260+0+0) & (0+0+0+2) \\ (0+0+2+0) & (0+0+2+0) & (0+0+6+0) & (0+0+0+0) \end{bmatrix}$$

$$= 13.36\times10^3 \begin{bmatrix} -8.741 & -2.223 & -2.741 & 1 \\ 1 & 1 & 3 & -3 \\ 9.262 & 3.786 & 3.262 & -3 \\ -3 & -3 & -9 & 3 \\ 0.260 & 0.78 & 0.260 & 2 \\ 2 & 2 & 6 & 0 \end{bmatrix} //$$