AU424 - Finite Element Methods and Analysis

Unit -2– ***General procedures of FEM***

Contents:

- ✓ Discretization
- ✓ Interpolation
- ✓ Shape Function
- ✓ Formulation of element characteristic matrices
- ✓ Assembly and solution

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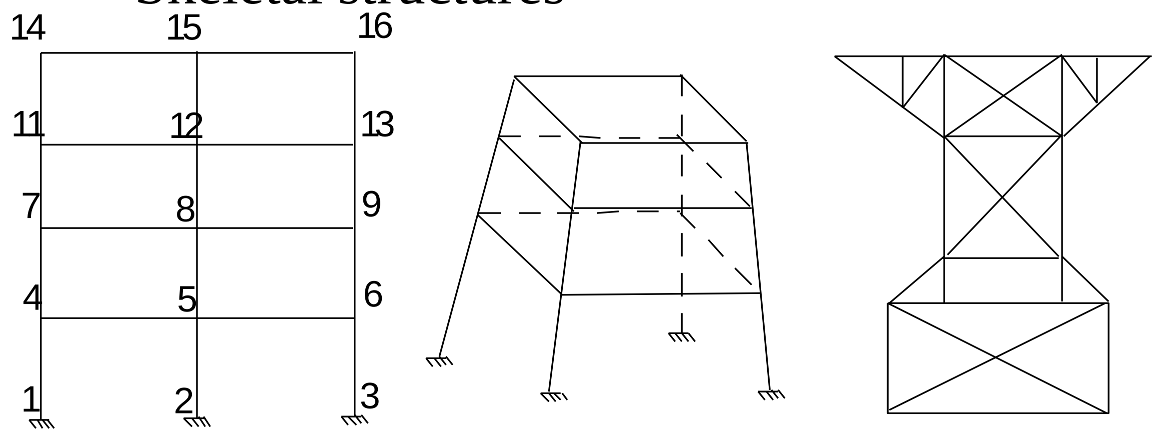
General Procedure for Finite Element Method

FEM is based on Direct Stiffness approach or Displacement approach.
A broad procedural outline is listed below

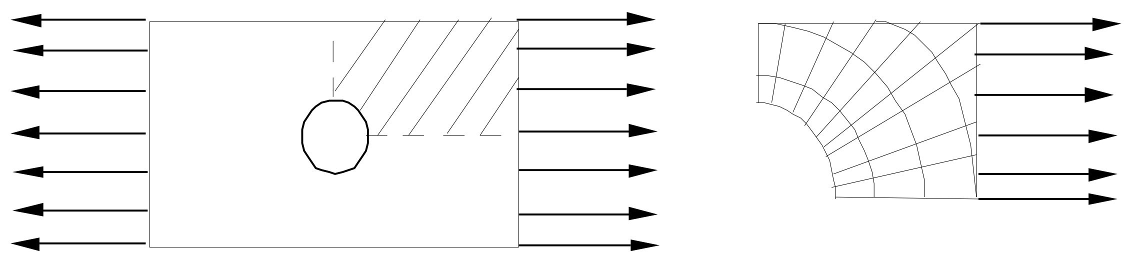
1. **Discretize and select element type.**

Skeletal structures

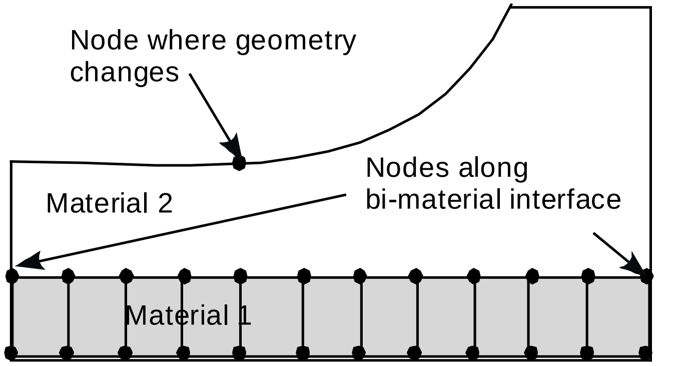
Skeletal structure gets discretized naturally. Member between two joints are treated as an element.



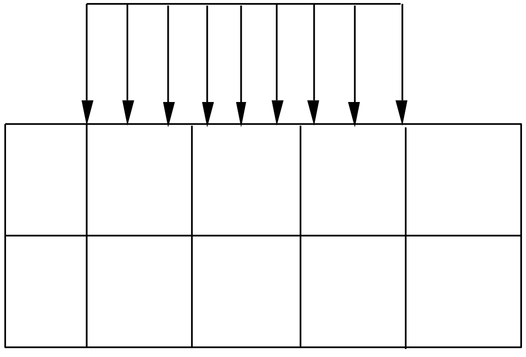
Continuums Arbitrary discretization.

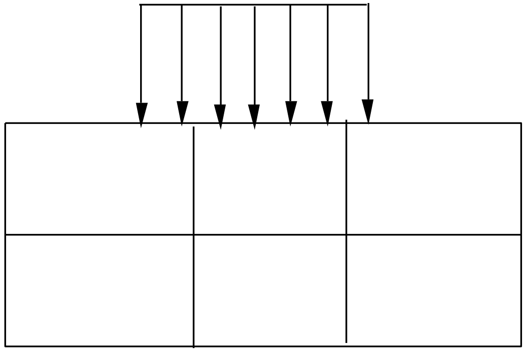


Precautions to be taken while discretization.



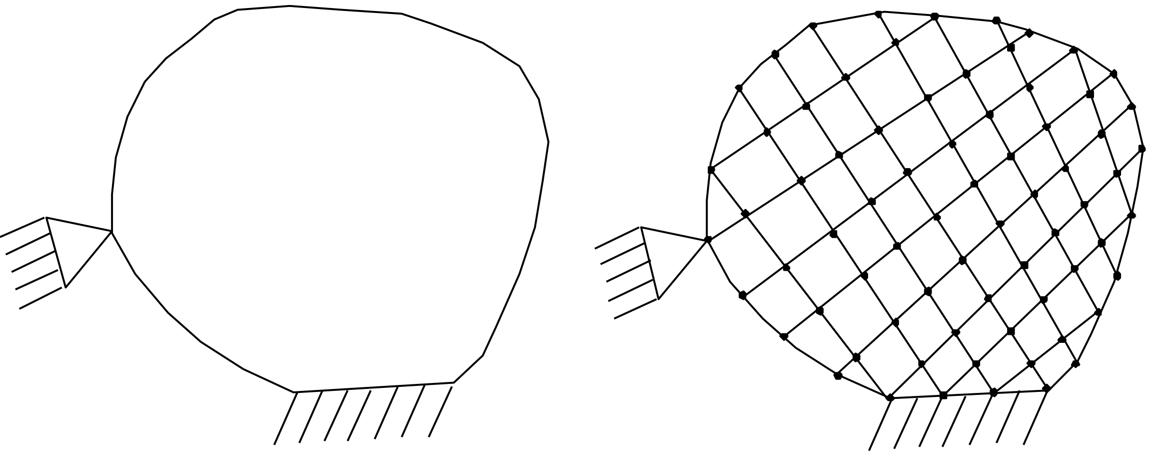
1.Provide nodes wherever geometry changes
2.Provide a set of nodes along bimaterial interface, so that no single element encompass or cover both materials. Element should cover one material.
3. Nodes at such points where concentrated load acts.
4.Nodes at points of specific interest for the analyst.
5.Nodes/|elements are provided such that distributed loads are covered completely by the element edge. Distributed load shall not be applied partially on any element edge.



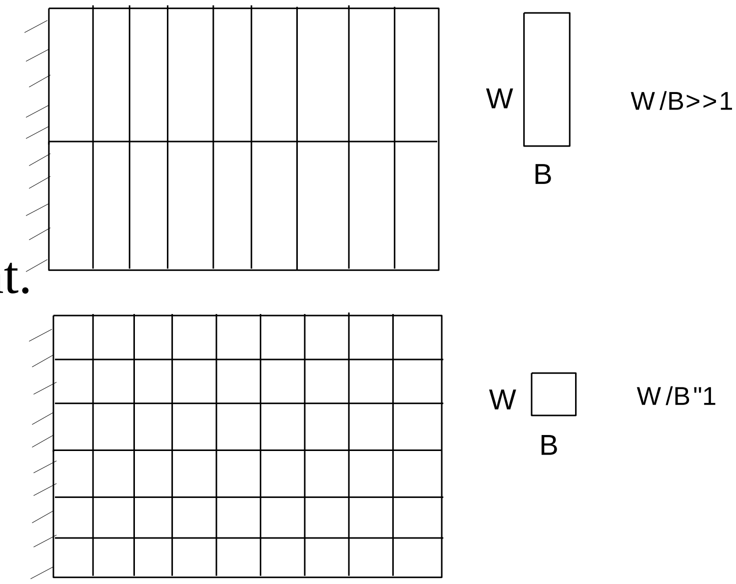


Precautions to be taken while discretization (contd)

6.Nodes to provide prescribed boundary condition.



7.Fine mesh in the regions of steep stress gradient.
8.Use symmetry condition
9 Aspect ratio of element≈1
10. Avoid obtuse/acute angle
11.Node numbering along shorter direction.



Fundamental concept of FEM

Consider a bar subjected to some exicitations like heating at one end. Let the field quantity flow through the body as fig2, which has been obtained by solving governing DE/PDE, In FEM the domain Ω is subdivided into subdomain and in each subdomain a piecewise continuous function is assumed.

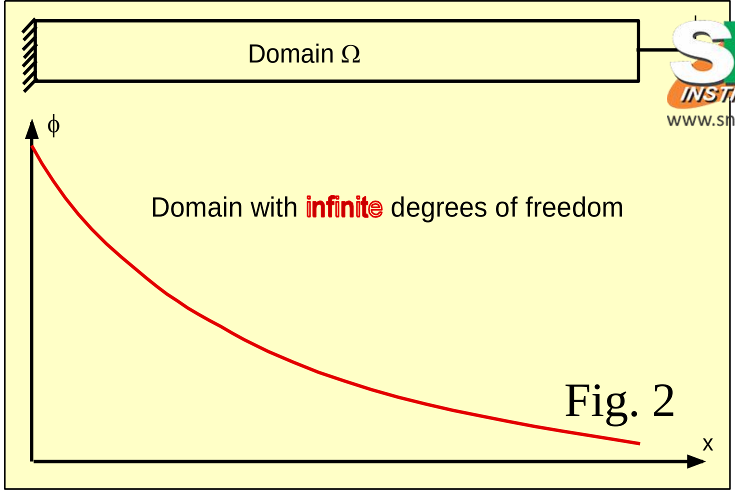
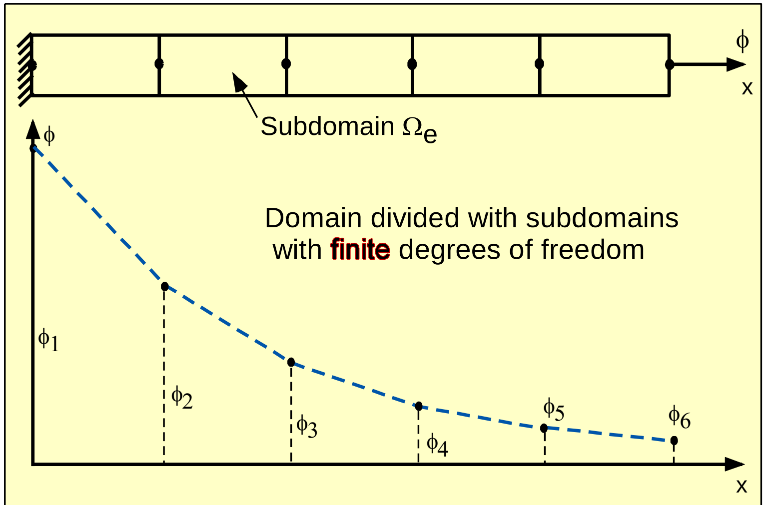
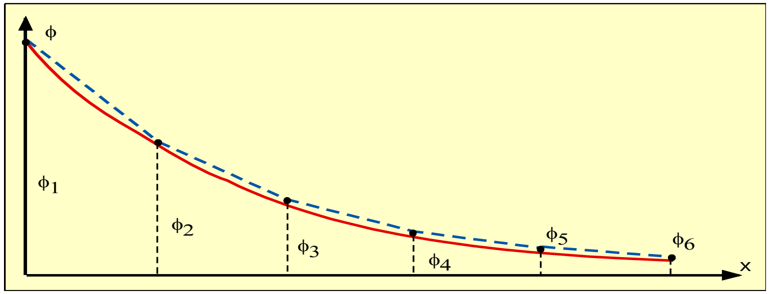


Fig. 2

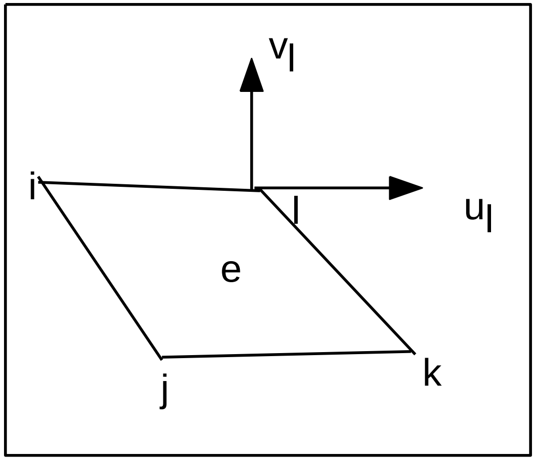


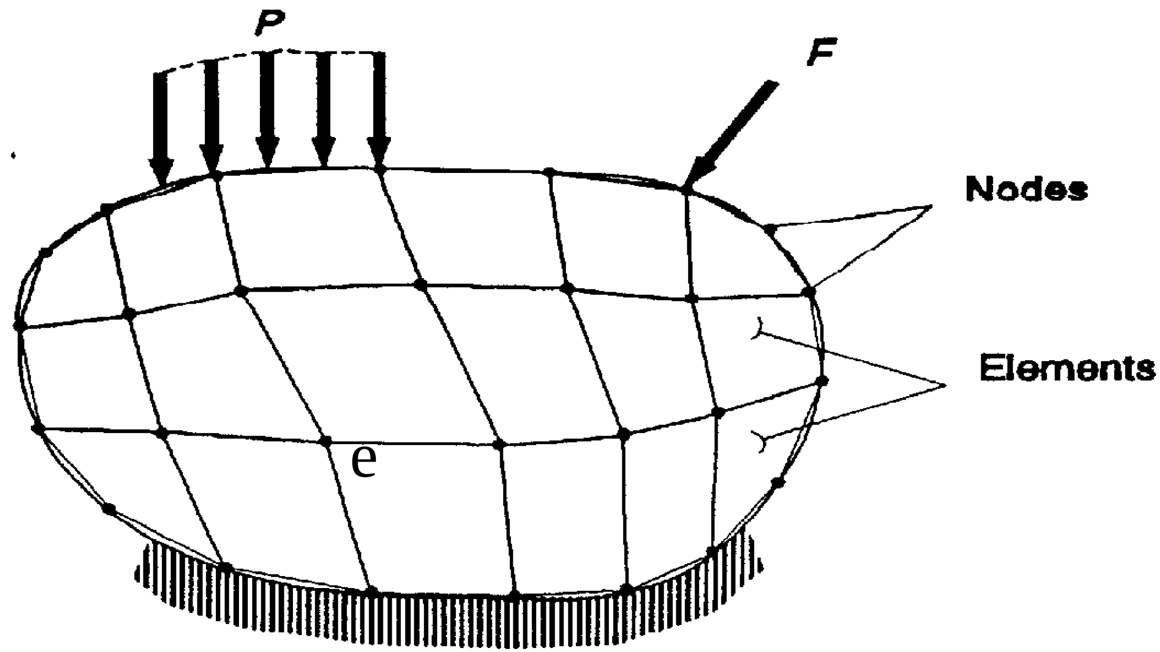


The fundamental concept of FEM is that continuous function of a continuum (given domain Ω) having infinite degrees of freedom is replaced by a discrete model, approximated by a set of piecewise continuous function having a finite degree of freedom.

Thus the method got the name finite element coined by Clough(1960).

2. Select displacement function



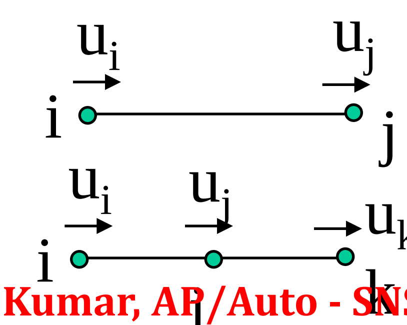


In the displacement approach a displacement function is assumed for the element. For example

For a one dimensional element

$$u(x) = \alpha_1 + \alpha_2 x$$

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$



For two dimensional rectangular elements displacement field at any interior of element is given by

$u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$

$v(x,y) = \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy$

Displacement at any interior point

$$\{\psi(x,y)\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}$$

Nodal displacement vector

$$\{d_i\} = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

1 —— Constant

$x \quad y$ —— Linear

$x^2 \quad xy \quad y^2$ —— Quadratic

$x^3 \quad x^2y \quad xy^2 \quad y^3$ —— Cubic

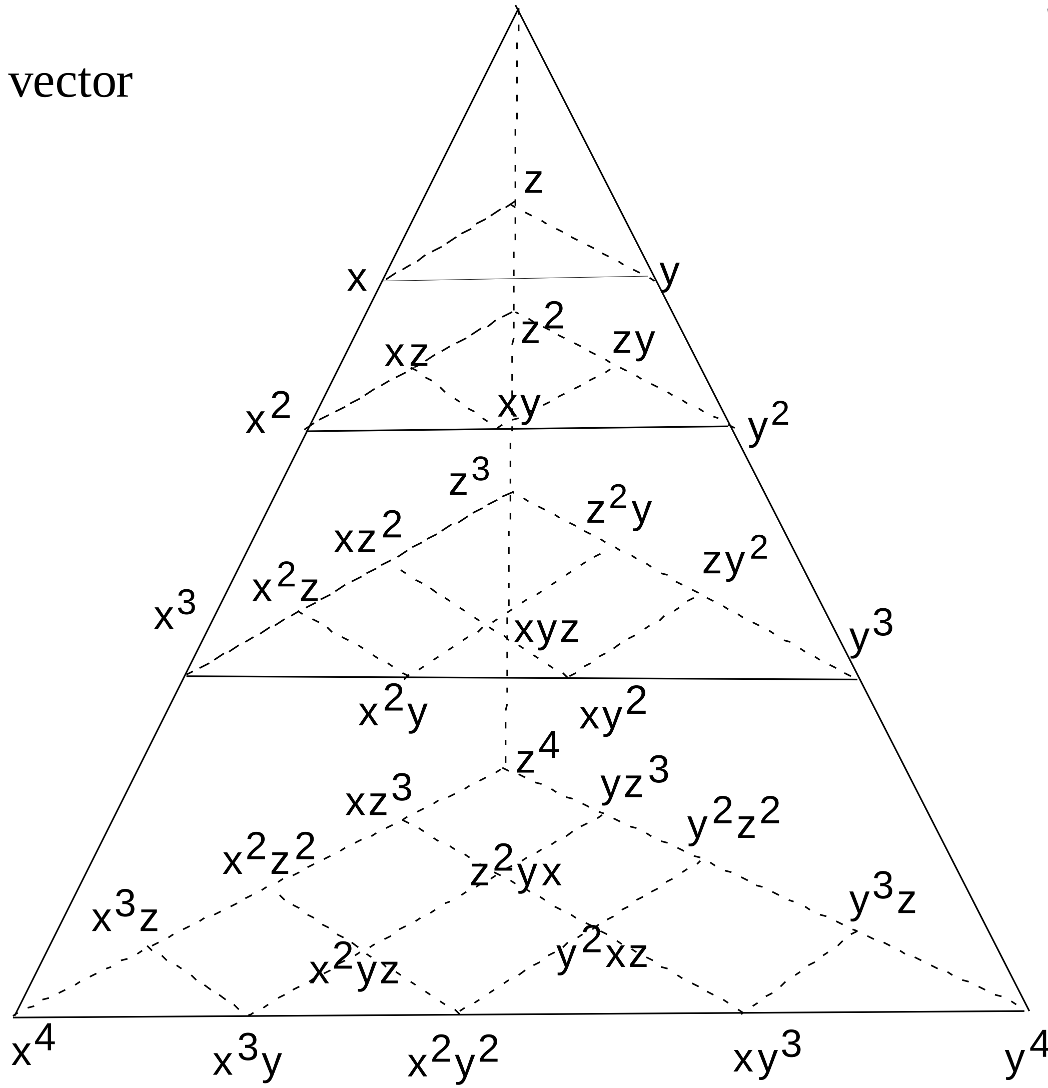
$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$ —— Quartic

Pascal triangle for 2D problems

The terms for displacement function is selected symmetrically from the pascal triangle to maintain geometric isotropy

Element displacement vector

$$\{d^e\} = \begin{Bmatrix} d_i \\ d_j \\ d_k \\ d_l \end{Bmatrix} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \\ u_l \\ v_l \end{Bmatrix}$$



Pascal triangle for 3D problems

$$u(x,y) = \begin{bmatrix} 1 & x & y & xy \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \qquad v(x,y) = \begin{bmatrix} 1 & x & y & xy \end{bmatrix} \begin{Bmatrix} \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{Bmatrix}$$

$$\{\psi(x,y)\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} 1 & x & y & xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & xy \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{Bmatrix} \quad (1a)$$

$$\{\psi(x,y)\} = \begin{bmatrix} 1 & x & y & xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & xy \end{bmatrix} \{\alpha\} \quad (1b)$$

Using the nodal conditions like

$$
\begin{aligned}
x = x_i, y = y_i &\Rightarrow u(x,y) = u_i, v(x,y) = v_i \\
x = x_j, y = y_j &\Rightarrow u(x,y) = u_j, v(x,y) = v_j \\
x = x_k, y = y_k &\Rightarrow u(x,y) = u_k, v(x,y) = v_k \\
x = x_l, y = y_l &\Rightarrow u(x,y) = u_l, v(x,y) = v_l
\end{aligned}
\qquad (2)
$$

This results in as many conditions as the number of unknown constants.

Using nodal boundary condition listed in eq. (2) in eq. 1a, following matrix eqn. Can be obtained

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_4 & y_4 & x_4 y_4 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{Bmatrix}$$

$$\{d\} = [A]\{\alpha\}$$

$$\{\alpha\} = [A]^{-1}\{d\} ------(3)$$

For quadrilateral element [A] is of size 8 X 8 substituting $\{\alpha\} = [A]^{-1}\{d\}$ in eq. 1b

$$\{\psi(x,y)\} = \begin{bmatrix} 1 & x & y & xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & xy \end{bmatrix}_{2X8} \underset{8X8}{[A]^{-1}} \{d\}$$

$$\{U(x,y)\} = \underset{2X8}{[N(x,y)]}\{d\}$$

$N(x,y) =$ is called displacement function

or interpolation function

or Shape function

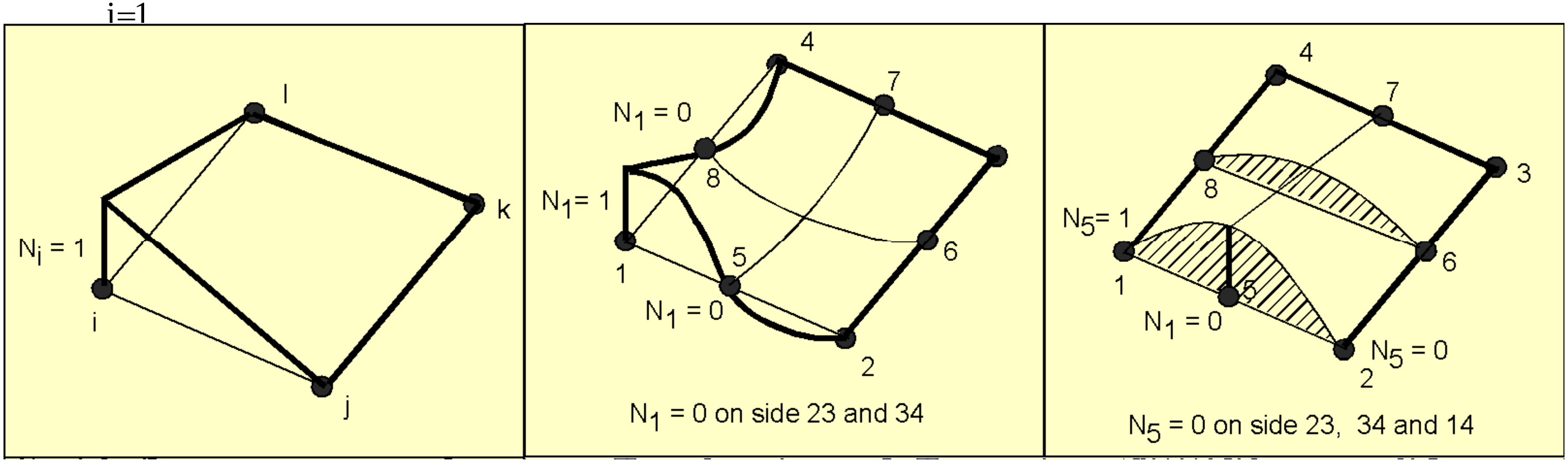
$$\{\psi(x,y)\} = \begin{bmatrix} N_i, & 0, & N_j, & 0, & N_k, & 0, & N_l, & 0 \\ 0, & N_i, & 0, & N_j, & 0, & N_k, & 0, & N_l \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ . \\ . \end{Bmatrix}$$

Properties of shape function

$N_i = 1.0$ at node 'i', and zero at all other nodes

$N_i = 0$ at all the sides on which node of interest does not fall.

$\sum_{i=1}^{n} N_i = 1.0$, n = number of nodes per element



3. Establish strain displacement and stress/strain relationship.

For one dimensional element

$$\varepsilon_x = \frac{du}{dx} = \left[\frac{d}{dx}\right]\{u(x)\}$$

For two dimensional element

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}, & 0 \\ 0, & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y}, & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$\{\varepsilon\} = [L]\{\psi(x,y)\} = \underset{3X2}{[L]}\underset{2X8}{[N(x,y)]}\{d\}$$

$$\{\varepsilon\} = \underset{3X8}{[B]}\{d\}$$

$$[B] == \begin{bmatrix} \frac{\partial N}{\partial x}, & 0 - - - - & \\ 0, & \frac{\partial N}{\partial y} - - - - & \\ \frac{\partial N}{\partial y}, & \frac{\partial N}{\partial x} - - - & \end{bmatrix}$$

For a linear elastic behavior the relationship between stresses and strains are of the form

$$\{\sigma\} = [D](\{\varepsilon\} - \{\varepsilon_0\}) + \{\sigma_0\}$$

$[D] =$ Elasticity matrix

$\{\varepsilon_0\} =$ initial strain vector (thermal strain αT)

$\{\sigma_0\} =$ initial residual stresses

4. Establish equilibrium equation to develop element stiffness relation.

Virtual work principle of a deformable body in equilibrium is subjected to arbitrary virtual displacement satisfying compatibility condition (admissible displacement), then the virtual work done by external(loads will be equal to virtual strain energy of internal stresses.

$$\delta U^e = \delta W^e$$

$$\text{Internal virtual energy } \delta U^e = \int_{v^e} \delta\{\varepsilon\}^T \{\sigma\} dv$$

$$\text{substitute } \{\sigma\} = [D](\{\varepsilon\} - \{\varepsilon_0\}) + \{\sigma_0\} \text{ in above eqn.}$$

$$\delta U^e = \int_{v^e} \delta\{\varepsilon\}^T \left([D](\{\varepsilon\} - \{\varepsilon_0\}) + \{\sigma_0\}\right) dv$$

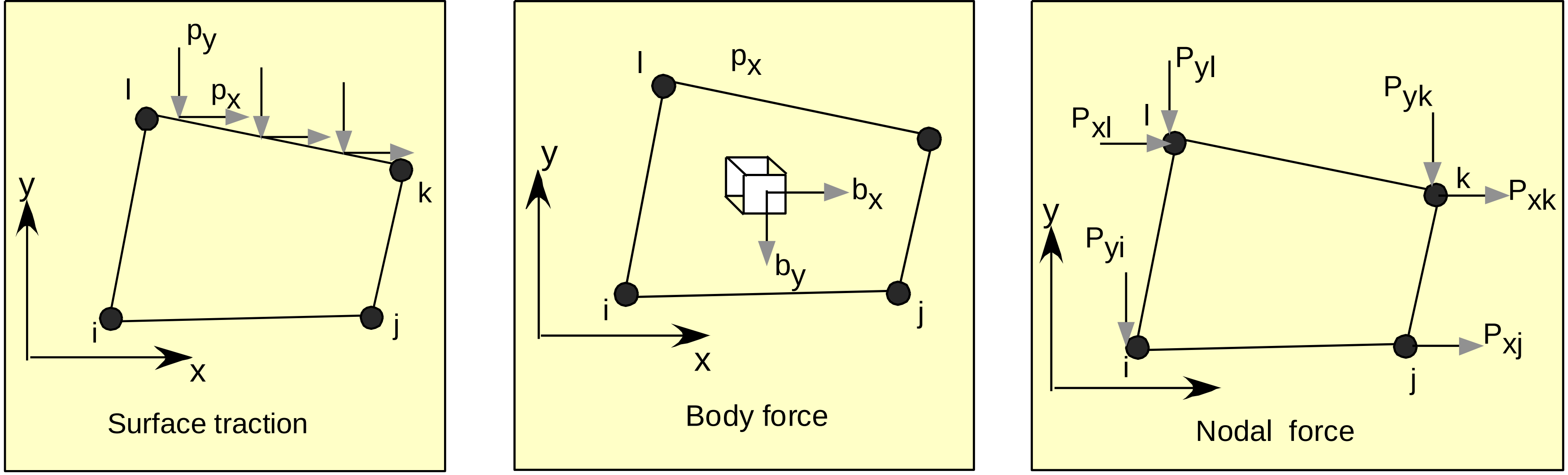
$$\delta U^e = \int_{v^e} \delta\{\varepsilon\}^T [D]\{\varepsilon\} dv - \int_{v^e} \delta\{\varepsilon\}^T [D]\{\varepsilon_0\} dv$$

$$+ \int_{v^e} \delta\{\varepsilon\}^T \{\sigma_0\} dv$$

$$\{\varepsilon\} = [B]\{d\}, \ \delta\{\varepsilon\} = [B]\delta\{d\}$$

$$\delta U^e = \int_{v^e} \delta\{d\}^T [B]^T [D][B]\{d\} dv$$

$$- \int_{v^e} \delta\{d\}^T [B]^T [D]\{\varepsilon_0\} dv + \int_{v^e} \delta\{d\}^T [B]^T \{\sigma_0\} dv$$



External virtual workdue to body force

$$\delta w_b^e = \int_{v^e} \delta\{\psi(x,y)\}^T \{b\} dv = \int_{v^e} \delta\{d\}^T [N]^T \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} dv$$

External virtual work due to surface force

$$\delta w_s^e = \int_{s} \delta\{\psi(x,y)\}^T \{p\} dv = \int_{s} \delta\{d\}^T [N]^T \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} dv$$

External virtual work due to nodal forces

$$\delta w_c^e = \delta\{d\}^T \{P^e\}, \quad \{P^e\}^T = \{P_{xi}, P_{yi}, P_{xj}, P_{yj}, \ldots\}$$

For equilibrium internal virtual work = external virtual work

$$\delta\{d\}^T\left(\int_{v^e}[B]^T[D][B]dv\{d\} - \int_{v^e}[B]^T[D]\{\varepsilon_0\}dv\right.$$

$$\left.+\int_{v^e}[B]^T\{\sigma_0\}dv\right) =$$

$$\delta\{d\}^T\left(\int_{v^e}[N]^T\begin{Bmatrix} b_x \\ b_y \end{Bmatrix}dv + \int_{s}[N]^T\begin{Bmatrix} p_x \\ p_y \end{Bmatrix}dv + \{P^e\}\right)$$

$$\Rightarrow [K_e]\{d^e\} = \{f_e\}$$

where

$$[K_e] = \int_{v^e}[B]^T[D][B]dv = \text{Element stiffness matrix}$$

$$\{f_e\} = \text{Total nodal force vector}$$

where

$$[K_e] = \int_{v^e} [B]^T [D][B] dv$$

$$\{f_e\} = \int_{v^e} [B]^T [D]\{\varepsilon_0\} dv - \int_{v^e} [B]^T \{\sigma_0\} dv)$$

$$\int_{v^e} [N]^T \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} dv + \int_{s} [N]^T \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} dv + \{P^e\}$$

First term in $\{f_e\}$ is equivalent nodal force vector due to initial strain. Second term is equivalent nodal force vector due to initial stress. Third term is equivalent nodal force vector due to body force. Fourth term is equivalent nodal force vector due to surface traction. Last term is applied concentrated load vector