3/11/19

Jacobians

If u and v are functions of two independent variables x & y, then the following determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$ is said to be Jacobian of u and v

with respect to x & y. It is denoted by

$$\frac{\partial(u,v)}{\partial(x,y)} \text{ or } \left[\frac{(u,v)}{(x,y)}\right] \text{ or } J$$

Property:

1. Jacobian of u, v, w with respect to x, y, z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Property: 2 The Jacobian of u, v with respect to x, y is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

3. If u and v are functions of x & y then $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$

4. If u, v are the functions of r, s where r, s are functions of x, y then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$

b. If u, v, w or w are functionally dependent functions of 3 independent variables x, y, z then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0 \text{ (or) } \frac{\partial(u,v)}{\partial(x,y)} = 0$$

Eg: 1

If $u = e^x \cos y$; $v = e^x \sin y$.

find $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$u = e^x \cos y$

$v = e^x \sin y$

$\frac{\partial u}{\partial x} = e^x \cos y$

$\frac{\partial v}{\partial x} = e^x \sin y$

$\frac{\partial u}{\partial y} = -e^x \sin y$

$\frac{\partial v}{\partial y} = e^x \cos y$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix}$$

$= (e^x \cdot e^x \cos^2 y + e^x e^x \sin^2 y)$

$= e^{2x} (\cos^2 y + \sin^2 y)$

$= e^{2x}$ //.

Eg: 2 find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ if $x = \frac{u^2}{v}$; $y = \frac{v^2}{w}$; $z = \frac{w^2}{u}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$x = u^2/v.$$

$$\frac{\partial x}{\partial u} = \frac{v(2u) - u^2 \frac{\partial v}{\partial u}}{v^2}$$

$$\frac{\partial x}{\partial u} = \frac{2uv - u^2 v}{v^2}$$

$$\frac{\partial x}{\partial v} = \frac{v(2u) - u^2}{v^2}$$

$$\frac{\partial x}{\partial v} = \frac{2uv - u^2}{v^2}$$

$$\frac{\partial x}{\partial w} = 0$$

$$x = u^2/v.$$

$$\frac{\partial x}{\partial u} = \frac{2u}{v}.$$

$$\frac{\partial x}{\partial v} = \frac{-u^2}{v^2}$$

$$\frac{\partial x}{\partial w} = 0$$

$$y = v^2/w$$

$$\frac{\partial y}{\partial u} = 0$$

$$\frac{\partial y}{\partial v} = \frac{2v}{w}$$

$$\frac{\partial y}{\partial w} = \frac{-v^2}{w^2}$$

$$\frac{w^2}{u} = z$$

$$\frac{\partial z}{\partial u} = \frac{-w^2}{u^2}$$

$$\frac{\partial z}{\partial v} = 0$$

$$\frac{\partial z}{\partial w} = \frac{2w}{u}.$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{2u}{v} & \frac{-u^2}{v^2} & 0 \\ 0 & \frac{2v}{w} & \frac{-v^2}{w^2} \\ \frac{-w^2}{u^2} & 0 & \frac{2w}{u} \end{vmatrix}$$

$$= \frac{2u}{v}\left(\left(\frac{2v}{w}\right)\left(\frac{2w}{u}\right) - 0\right) + \frac{u^2}{v^2}\left(0 - \frac{v^2}{w^2}\left(\frac{w^2}{u^2}\right)\right) + 0$$

$$= \frac{2u}{v}\left(\frac{4v}{u}\right) + \frac{u^2}{v^2}\left(-\frac{v^2}{u^2}\right).$$

$$= 8 - 1$$

$$= 7.$$

Eg:3

ST: the function $u = x/y$, $v = \frac{x+y}{x-y}$ are functionally dependent.

$$\frac{\partial(u,v)}{\partial(x,y)} = 0$$

$u = x/y$

$$\frac{\partial u}{\partial x} = \frac{1}{y}$$

$$\frac{\partial u}{\partial y} = \frac{-x}{y^2}$$

$v = \frac{x+y}{x-y}$

$$\frac{\partial v}{\partial x} = \frac{(x-y)(1+\cancel{0}) - (x+y)(1-0)}{(x-y)^2}$$

$$\frac{\partial v}{\partial x} = \frac{(x-y)-(x+y)}{(x-y)^2}$$

$$= \frac{x-y-x-y}{(x-y)^2}$$

$$= \frac{-2y}{(x-y)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x-y)(0+1) - (x+y)(0-1)}{(x-y)^2}$$

$$= \frac{(x-y)+(x+y)}{(x-y)^2}$$

$$= \frac{x-y+x+y}{(x-y)^2}$$

$$\frac{\partial v}{\partial y} = \frac{2x}{(x-y)^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{y} & \frac{-x}{y^2} \\ \frac{-2y}{(x-y)^2} & \frac{2x}{(x-y)^2} \end{vmatrix}$$

$$= \frac{1}{y}\left(\frac{2x}{(x-y)^2}\right) - \left(\frac{-x}{y^2}\right)\left(\frac{-2y}{(x-y)^2}\right)$$

$$= \frac{2x}{y(x-y)^2} - \frac{2xy}{y(x-y)^2} = 0$$

$\therefore$ This is linearly dependent

Notes:

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

Eg:

If $u = xy$, $v = x^2 - y^2$ where $x = r\cos\theta$ and $y = r\sin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$.

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

$u = xy$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$\frac{\partial u}{\partial x} = y$

$\frac{\partial u}{\partial y} = x$

$v = x^2 - y^2$

$\frac{\partial v}{\partial x} = 2x$

$\frac{\partial v}{\partial y} = -2y$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ 2x & -2y \end{vmatrix} = y(-2y) - x(2x)$$

$$= -2y^2 - 2x^2$$

$$= -2(y^2 + x^2)$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$x = r\cos\theta$; $y = r\sin\theta$.

$\frac{\partial x}{\partial r} = \cos\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial y}{\partial \theta} = r\cos\theta$.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r.$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = -2(x^2+y^2)(r)$$

$$= -2r(r^2\cos^2\theta + r^2\sin^2\theta)$$

$$= -2r^3(\cos^2\theta + \sin^2\theta)$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = -2r^3.$$