4/11/19.

UNIT - IV Functions of Several Variables

Partial differentiation:

Let $u = f(x, y)$ be a functions of two independent variables, differentiation u with respect to 'x', keeping y as constant is known as the partial differential co. eff of u with respect to 'x'.

u is denoted by $\frac{\partial u}{\partial x}$

Notes:

$\frac{\partial u}{\partial x}$ - means differentiated u with respect to x. keeping y as constant.

$\frac{\partial u}{\partial y}$ - means differentiate u with respect to y keeping x as constant.

* $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$

* $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$

* $\frac{\partial^2 u}{\partial y . \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$

* $\frac{\partial^2 u}{\partial x . \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

* $\frac{\partial^2 u}{\partial x . \partial y} = \frac{\partial^2 u}{\partial y . \partial x}$.

1. Find the first and second derivative of $u = x^3 + y^3 - 3axy$

$$\frac{\partial u}{\partial x} = 3x^2 - 3ay \qquad \frac{\partial u}{\partial y} = 3y^2 - 3ax$$

$$\frac{\partial^2 u}{\partial x^2} = 6x \qquad \frac{\partial^2 u}{\partial y^2} = 6y$$

$$\frac{\partial^2 u}{\partial y \cdot \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \qquad \frac{\partial^2 u}{\partial x \cdot \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}(3x^2 - 3ay) \qquad = \frac{\partial}{\partial x}(3y^2 - 3ax)$$

$$= -3a \qquad = -3a$$

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2. If $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$ then show that,

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}$$

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \rightarrow ①$$

Diff eqn ① w.r. to 'x' partially

$$\cancel{2}(x-a) = \cancel{2}r \cdot \frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{x-a}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r(1) - (x-a)\cdot\frac{\partial r}{\partial x}}{r^2} = \frac{r - \frac{(x-a)^2}{r}}{r^2}$$

$$= \frac{\frac{r^2 - (x-a)^2}{r}}{r^2} = \frac{r^2 - (x-a)^2}{r^3}$$

$$= \frac{\cancel{r^2}}{\cancel{r^3}} - \frac{(x-a)^2}{r^3} = \frac{1}{r} - \frac{(x-a)^2}{r^3} \rightarrow ②$$

III ly. $\frac{\partial r}{\partial y} = \frac{(y-b)}{r}$

$$\frac{\partial^2 r}{\partial y^2} = \frac{1}{r} - \frac{(y-b)^2}{r^3} \rightarrow ③$$

III ly. $\frac{\partial r}{\partial z} = \frac{(z-c)}{r}$

$$\frac{\partial^2 r}{\partial z^2} = \frac{1}{r} - \frac{(z-c)^2}{r^3} \rightarrow ④ .$$

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{1}{r} - \frac{(x-a)^2}{r^3} + \frac{1}{r} - \frac{(y-b)^2}{r^3} + \frac{1}{r} - \frac{(z-c)^2}{r^3}$$

$$= \frac{r^2-(x-a)^2}{r^3} + \frac{r^2-(y-b)^2}{r^3} + \frac{r^2-(z-c)^2}{r^3}$$

$$= \frac{3r^2 - ((x-a)^2+(y-b)^2+(z-c)^2)}{r^3}$$

$$= \frac{3r^2 - r^2}{r^3} = \frac{2\cancel{r^2}}{r^{\cancel{3}}}$$

$$= \frac{2}{r}$$

∴ Hence verified.

If $z = (x^2+xy+y^2)^r$ then S.T :

$$\left(x\frac{\partial}{\partial x} + y\cdot\frac{\partial}{\partial y}\right)\left(x\cdot\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y}\right) = 4zr^2.$$

given:

$$z = (x^2+xy+y^2)^r \rightarrow ①$$

Diff ① w.r.to x, y, z

$$\frac{\partial z}{\partial x} = r(x^2+xy+y^2)^{r-1}\cdot(2x+y)$$

$$x\cdot\frac{\partial z}{\partial x} = xr(2x+y)(x^2+xy+y^2)^{r-1}$$

$$\frac{\partial z}{\partial y} = r(x^2+xy+y^2)^{r-1} \cdot (x+2y)$$

$$y \cdot \frac{\partial z}{\partial y} = yr(x+2y)(x^2+xy+y^2)^{r-1}$$

$$x \cdot \left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial x}\right) = xr(2x+y)(x^2+xy+y^2)^{r-1} + yr(x+2y)(x^2+xy+y^2)^{r-1}$$

$$= r(x^2+xy+y^2)^{r-1} \cdot (x(2x+y)+y(x+2y))$$

$$= r(x^2+xy+y^2)^{r-1} \cdot (2x^2+xy+xy+2y^2)$$

$$= r(x^2+xy+y^2)^{r-1} \cdot (2x^2+2xy+2y^2)$$

$$= 2r(x^2+xy+y^2)^{r-1} \cdot (x^2+xy+y^2)$$

$$= 2r(x^2+xy+y^2)^{r}$$

$$= 2rz.$$

$$\left(x \cdot \frac{\partial}{\partial x} + y \cdot \frac{\partial}{\partial y}\right)\left(x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y}\right) = x \cdot \frac{\partial}{\partial x} + y \cdot \frac{\partial}{\partial y}(2rz)$$

$$= x \cdot \frac{\partial}{\partial x}(2rz) + y \cdot \frac{\partial}{\partial y}(2rz)$$

$$= 2r\left(\frac{\partial z}{\partial x} \cdot x + y \cdot \frac{\partial z}{\partial y}\right)$$

$$= 2r(2rz)$$

$$= 4zr^2.$$

$\therefore$ Hence verified.

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verify that $U_{xy} = U_{yx}$ when $u = \tan^{-1}\left(\frac{x}{y}\right)$

given,

$$u = \tan^{-1}\left(\frac{x}{y}\right) \rightarrow ①$$

Diff ① w.r to x,

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(\frac{1}{y}\right)$$

$$= \frac{y^2}{y^2+x^2} \cdot \left(\frac{1}{y}\right)$$

$$= \frac{y}{x^2+y^2} \longrightarrow ②$$

Diff ② w.r.to y,

$$\frac{\partial^2 u}{\partial y \cdot \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{x^2+y^2(1) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$U_{yx} = \frac{x^2-y^2}{(x^2+y^2)^2} \longrightarrow ③$$

Diff ① w.r.to 'y',

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(\frac{-x}{y^2}\right)$$

$$= \frac{y^2}{y^2+x^2} \cdot \left(\frac{-x}{\cancel{y^2}}\right) = \frac{-x}{y^2+x^2} \longrightarrow ④$$

Diff ④ w.r to 'x'.

$$\frac{\partial^2 u}{\partial x \cdot \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right)$$

$$= -\left[\frac{x^2+y^2(1) - x(2x)}{(x^2+y^2)^2}\right]$$

$$= -\left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right] = -\left[\frac{-x^2+y^2}{(x^2+y^2)^2}\right]$$

$$U_{xy} = \frac{x^2-y^2}{(x^2+y^2)^2} \longrightarrow ⑤$$

From 3 & 5,

$$V_{yx} = V_{xy}$$