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Envelope :

Definition :

A curve which touches each member of a family of curves is called envelope.

Type : 1

Note: The envelope of the family of curves $A\lambda^2 + B\lambda + C = 0$ when A, B, C are functions of x and y is $B^2 - 4AC = 0$.

Find the envelope of $y = mx + \frac{a}{m}$, m being the parameter.

$$y = mx + \frac{a}{m}$$

$$my = m^2x + a$$

$$m^2x - my + a = 0$$

Ex: 1
$x\cos\theta + y\sin\theta = a$ — (1)
where θ is a parameter.

which is the quadratic eqn with parameter 'm'.

$$Ax^2 + B\cancel{y}^{x} + C = 0$$

Diff
— (2)
square and add
$x^2 + y^2 = a^2$.

$$B^2 - 4AC = 0$$

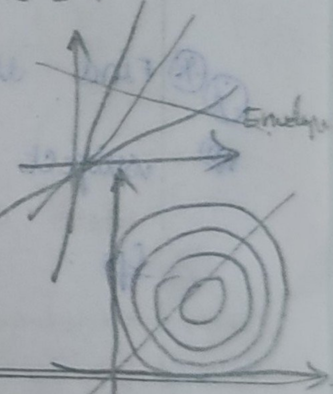
$$A = x,\ B = -y,\ C = a$$

$y = mx$
$x = 1 \Rightarrow y = mx$
$x = 2 \Rightarrow y = 2x$
$x = 3 \Rightarrow y = 3x$

$$B^2 - 4AC = 0$$

$$y^2 - 4ax = 0$$

which is the envelope



Find the envelope of $y = mx + \sqrt{a^2m^2 - b^2}$, m parameter

$$y = mx + \sqrt{a^2m^2 - b^2}$$

$$y - mx = \sqrt{a^2m^2 - b^2}$$

Ex: 3 $y = mx + m^3$
Diff p.w.r.t m
$0 = x + 3m^2$
$m^2 = -\frac{x}{3}$
$y = mx + m^3$
$= m(x + m^2)$
$= m(x - \frac{x}{3})$

sq. on b.s,

$$(y - mx)^2 = a^2m^2 - b^2$$

$$y^2 + m^2x^2 - 2yxm = a^2m^2 - b^2$$

$$y^2 + m^2x^2 - 2yxm - a^2m^2 + b^2 = 0$$

$$m^2x^2 - a^2m^2 - 2yxm + y^2 + b^2 = 0$$

$$m^2(x^2 - a^2) - 2yxm + (y^2 + b^2) = 0$$

$A = x^2 - a^2$, $B = -2yx$, $C = (y^2 + b^2)$

$$B^2 - 4AC = 0$$

$$4x^2y^2 - 4(x^2 - a^2)(y^2 + b^2) = 0$$

$$4x^2y^2 - 4(x^2y^2 + x^2b^2 - a^2y^2 - a^2b^2) = 0 \quad [\div 4]$$

$$\cancel{x^2y^2} - \cancel{x^2y^2} - x^2b^2 + a^2y^2 + a^2b^2 = 0$$

$$-x^2b^2 + a^2y^2 = -a^2b^2 \quad [\div -a^2b^2]$$

$$\frac{x^2}{a^2} - \frac{y^2}{b} = 1$$

which is the envelope.

Envelope of two parameters of family of curves

4m ⓧ Find the envelope of the st. line $\frac{x}{a} + \frac{y}{b} = 1$ subject to the condition $ab = c^2$, c is constant.

where a & b are parameter

Sln: $\frac{x}{a} + \frac{y}{b} = 1 \rightarrow ①$

$ab = c^2 \rightarrow ②$

From ② $b = c^2/a$ sub in ①

$$\frac{x}{a} + \frac{y}{c^2/a} = 1$$

$$\frac{xc^2 + a^2y}{ac^2} = 1$$

$$\frac{xc^2 + a^2y}{ac^2} = 1$$

$$xc^2 + a^2y = ac^2$$

$$a^2y - ac^2 + xc^2 = 0 \Rightarrow Ax^2 + Bx + C = 0$$

$$ya^2 - c^2a + xc^2 = 0$$

where, $A = y, B = -c^2; C = xc^2$

$$B^2 = 4AC$$

$$(-c^2)^2 = 4yxc^2$$

$$c^4 = 4xyc^2$$

$$c^2 = 4xy$$

Find the envelope of $\frac{x}{a} + \frac{y}{b} = 1$ subject to $a + b = c$, here c is a constant, where a & b are parameter

Sln: $\frac{x}{a} + \frac{y}{b} = 1 \rightarrow ①$

$a + b = c \rightarrow ②$

Diff ① w.r. to 'a'.

$$x\left(\frac{-1}{a^2}\right) + y\left(\frac{-1}{b^2}\right) \cdot \frac{db}{da} = 0$$

$$-\frac{x}{a^2} - \frac{y}{b^2} \cdot \frac{db}{da} = 0$$

$$-\frac{y}{b^2} \cdot \frac{db}{da} = \frac{x}{a^2}$$

$$\frac{db}{da} = -\frac{xb^2}{ya^2} \rightarrow ③$$

Diff ② w.r.to 'a'

$$1 + \frac{db}{da} = 0$$

$$\frac{db}{da} = -1 \rightarrow ④$$

Equate ③ & ④,

$$+\frac{xb^2}{ya^2} = +1$$

$$\frac{x}{a^2} = \frac{y}{b^2}$$

$$\frac{\frac{x}{a}}{a} = \frac{y/b}{b}$$

$$\frac{x}{a^2} = \frac{y}{b^2} = \frac{\frac{x}{a} + \frac{y}{b}}{a+b}$$

$$\frac{x}{a^2} = \frac{y}{b^2} = \frac{1}{c}$$

$$\frac{x}{a^2} = \frac{1}{c}$$

$$a^2 = xc$$

$$a = (xc)^{1/2}$$

$$\frac{y}{b^2} = \frac{1}{c}$$

$$b^2 = yc$$

$$b = (yc)^{1/2}$$

sub in ②,

$$(xc)^{1/2} + (yc)^{1/2} = c$$

$$c^{1/2}(x^{1/2} + y^{1/2}) = c$$

$$x^{1/2} + y^{1/2} = c^{1/2}$$

(*) ⑧ Find the envelope of $\frac{x}{a} + \frac{y}{b} = 1$, subject to $a^n + b^n = c^n$, where c is constant. where a & b parameter

IMP

$$\frac{x}{a} + \frac{y}{b} = 1 \rightarrow (1)$$

$$a^n + b^n = c^n \rightarrow (2)$$

Diff (1) w.r. to 'a'.

$$x\left(\frac{-1}{a^2}\right) + y\left(\frac{-1}{b^2}\right) \cdot \frac{db}{da} = 0$$

$$\frac{-y}{b^2} \cdot \frac{db}{da} = \frac{x}{a^2}$$

$$\frac{db}{da} = \frac{-xb^2}{a^2 y} \rightarrow (3)$$

Diff (2) w.r. to 'a'

$$na^{n-1} + nb^{n-1} \cdot \frac{db}{da} = 0$$

$$nb^{n-1} \cdot \frac{db}{da} = -na^{n-1}$$

$$\frac{db}{da} = \frac{-a^{n-1}}{b^{n-1}} \rightarrow (4)$$

Equate (3) & (4),

$$\frac{-xb^2}{ya^2} = \frac{-a^{n-1}}{b^{n-1}}$$

$$\frac{x}{a^{n-1} \cdot a^2} = \frac{y}{b^{n-1} \cdot b^2}$$

$$\frac{x}{a^{n+1}} = \frac{y}{b^{n+1}}$$

i.e.,

$$\frac{\frac{x}{a}}{a^n} = \frac{\frac{y}{b}}{b^n}$$

$$\frac{\frac{x}{a} + \frac{y}{b}}{a^n + b^n} = \frac{1}{c^n}$$

$$\frac{x}{a^{n+1}} = \frac{y}{b^{n+1}} = \frac{1}{c^n}$$

$$\frac{x}{a^{n+1}} = \frac{1}{c^n} \qquad \frac{y}{b^{n+1}} = \frac{1}{c^n}.$$

$$a^{n+1} = xc^n \qquad b^{n+1} = yc^n.$$

÷ b. s by $\frac{n}{n+1}$

$$a^n = (xc^n)^{\frac{n}{n+1}} \qquad b^n = (yc^n)^{n/n+1}$$

sub in ②,

$$(xc^n)^{n/n+1} + (yc^n)^{n/n+1} = c^n.$$

$n - \frac{n^2}{n+1}$

$\frac{n^2+n-n^2}{n+1}$

$$c^{\frac{n^2}{n+1}}\left(x^{\frac{n}{n+1}} + y^{n/n+1}\right) = c^n.$$

$$x^{n/n+1} + y^{n/n+1} = c^{n - n^2/n+1}$$

$\frac{n^2 = n - n^2}{n+1}$

$$x^{n/n+1} + y^{n/n+1} = c^{n/n+1}$$

Find the envelope of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ subject to $a+b=c$, c is a constant.

Sol:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow ①$$

$$a+b = c \rightarrow ②$$

Diff ① w.r to 'a',

$$x^2\left(\frac{-2}{a^3}\right) + y^2\left(\frac{-2}{b^3}\right)\cdot\frac{db}{da} = 0$$

$$-\frac{\cancel{2}y^2}{b^3}\cdot\frac{db}{da} = \frac{\cancel{2}x^2}{a^3}$$

$$\frac{db}{da} = -\frac{x^2b^3}{y^2a^3} \rightarrow ③$$

Diff ② w.r to 'a'

$$1 + \frac{db}{da} = 0$$

$$\frac{db}{da} = -1 \rightarrow ④$$

Equate ③ & ④,

$$\frac{\cancel{-}x^2b^3}{y^2a^3} = \cancel{-}1$$

$$\frac{x^2}{a^3} = \frac{y^2}{b^3}$$

$$\frac{\frac{x^2}{a^2}}{a} = \frac{\frac{y^2}{b^2}}{b}$$

$$\frac{\frac{x^2}{a^2} + \frac{y^2}{b^2}}{a+b} = \frac{1}{c}$$

$$\frac{x^2}{a^3} = \frac{1}{c}$$

$$a^3 = cx^2$$

$$a = (cx^2)^{1/3}$$

$$\frac{y^2}{b^3} = \frac{1}{c}$$

$$b^3 = y^2 c$$

$$b = (cy^2)^{1/3}$$

sub in ②,

$$a + b = c$$

$$(cx^2)^{1/3} + (cy^2)^{1/3} = c$$

$$c^{1/3} \cdot x^{2/3} + c^{1/3} \cdot y^{2/3} = c$$

$$c^{1/3}(x^{2/3} + y^{2/3}) = c \Rightarrow x^{2/3} + y^{2/3} = c/c^{1/3}$$

$$x^{2/3} + y^{2/3} = c^{2/3}$$

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⑤ Find the envelope of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ subject to $a^n + b^n = c^n$, c is constant.

Sol:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow ①$$

$$a^n + b^n = c^n \rightarrow ②$$

Diff ① w.r to 'a'

$$x^2\left(\frac{-2}{a^3}\right) + y^2\left(\frac{-2}{b^3}\right)\frac{db}{da} = 0$$

$$\frac{-2y^2}{b^3} \cdot \frac{db}{da} = \frac{2x^2}{a^3}$$

$$\frac{db}{da} = -\frac{x^2 b^3}{y^2 a^3} \rightarrow ③$$

Diff ② w.r to 'a',

$$1 + \frac{db}{da} = 0$$

$$na^{n-1} + nb^{n-1} \cdot \frac{db}{da} = 0$$

$$\cancel{n}b^{n-1} \cdot \frac{db}{da} = -\cancel{n}a^{n-1}$$

$$\frac{db}{da} = -\frac{a^{n-1}}{b^{n-1}} \rightarrow (4)$$

Equate (3) & (4),

$$\frac{\cancel{-}x^2b^3}{y^2a^3} = \frac{\cancel{-}a^{n-1}}{b^{n-1}}$$

$$\frac{x^2}{a^3 \cdot a^{n-1}} = \frac{y^2}{b^3 \cdot b^{n-1}} \Rightarrow \frac{x^2}{a^{n+2}} = \frac{y^2}{b^{n+2}}$$

$$\frac{\frac{x^2}{a^2}}{a^n} = \frac{\frac{y^2}{b^2}}{b^n} \Rightarrow \frac{\frac{x^2}{a^2} + \frac{y^2}{b^2}}{a^n + b^n} = \frac{1}{c^n}$$

$$\frac{x^2}{a^{n+2}} = \frac{y^2}{b^{n+2}} = \frac{1}{c^n}$$

$$\frac{x^2}{a^{n+2}} = \frac{1}{c^n} \qquad \frac{y^2}{b^{n+2}} = \frac{1}{c^n}$$

$$a^{n+2} = x^2c^n \qquad b^{n+2} = c^ny^2$$

$$a^n = (x^2 \cdot c^n)^{n/n+2} \qquad b^n = (c^n y^2)^{n/n+2}$$

$$(x^2c^n)^{n/n+2} + (y^2c^n)^{n/n+2} = c^n$$

$$x^{2n/n+2} \cdot c^{n^2/n+2} + y^{2n/n+2} c^{n^2/n+2} = c^n$$

$$c^{n^2/n+2}\left(x^{2n/n+2} + y^{2n/n+2}\right) = c^n$$

$$x^{2n/n+2} + y^{2n/n+2} = c^{2n/n+2}$$