UNIT - V

SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

Second order linear differential equation with Constant Coefficients

Consider a second order linear differential equation is,

$$(a_0 D^2 + a_1 D + a_2) y = R(x)$$

To find complementary function :

The auxiliary equation is,

$$a_0 m^2 + a_1 m + a_2 = 0.$$

Nature of roots	Complementary function (C.F)
(1) m_1 & m_2 are real and different	$C.F = Ae^{m_1 x} + Be^{m_2 x}$
(2) m_1 and m_2 are real and equal	$C.F = (Ax+B)e^{mx}$
(3) m_1 and m_2 are complex, let $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$	$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

Problems to find complementary function :

1. Solve : $(D^2 - 5D + 6)y = 0$

Soln : The A.E is

$$m^2 - 5m + 6 = 0$$

$$m = 2, 3$$

The roots are real & different.

$$C.F = Ae^{m_1 x} + Be^{m_2 x}$$

$$C.F = Ae^{2x} + Be^{3x}$$

$$\therefore \boxed{y = Ae^{2x} + Be^{3x}}$$

2. Solve : $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 0$

Soln ;

Given : $(D^2 - 6D + 9)y = 0$

The A.E is

$$m^2 - 6m + 9 = 0$$

$$m = 3, 3$$

The roots are real and equal.

$$C.F = (Ax + B)e^{mx}$$

$$= (Ax + B)e^{3x}$$

$$\therefore \boxed{y = (Ax + B)e^{3x}}$$

Rules to find particular integral (P.I):

$$P.I = \frac{1}{f(D)} R(x)$$

Failure case: When $f(D) = 0$.

$$\Rightarrow P.I = \frac{x}{f'(D)} R(x)$$

Again failure, When $f'(D) = 0$

$$P.I = \frac{x^2}{f''(D)} R(x)$$

and so on.

$$\boxed{\begin{array}{l}\text{Type I : } R(x) = e^{ax} \\ \text{Replace } D \to a\end{array}}$$

1. Solve $(D^2+1)y = e^{-x}$

Soln: The A.E is $m^2+1 = 0 \Rightarrow m^2 = -1$

$$m = \pm i = 0 \pm i$$

$$\alpha = 0, \beta = 1$$

$$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

$$= e^{0x}(A\cos x + B\sin x).$$

$$C.F = A\cos x + B\sin x$$

$$P.I = \frac{e^{-x}}{D^2+1}$$

$e^{ax} = e^{-x}$

Here $a = -1$

$D \rightarrow a \rightarrow -1$

$$= \frac{e^{-x}}{(-1)^2+1} = \frac{e^{-x}}{2}$$

$$y = C.F + P.I$$

$$\boxed{y = A\cos x + B\sin x + \frac{e^{-x}}{2}}$$

(2) Solve : $(D^2+4D+4)y = 11e^{-2x}$

Soln : The A.E is

$$m^2+4m+4 = 0$$

$$(m+2)^2 = 0$$

$$m = -2, -2$$

The roots are real and equal.

$$C.F = (Ax+B)\,e^{mx}$$

$$C.F = (Ax+B)\,e^{-2x}$$

$$P.I = \frac{11e^{-2x}}{D^2+4D+4}$$

$e^{ax} = e^{-2x}$

$a = -2$

$D \rightarrow a \rightarrow -2$

$$= \frac{11e^{-2x}}{(-2)^2+4(-2)+4}$$

$$= \frac{11e^{-2x}}{4-8+4} = \frac{11e^{-2x}}{0}$$ (failure case)

$$= \frac{11x e^{-2x}}{2D+4}$$

$$= \frac{11xe^{-2x}}{2(-2)+4} = \frac{11xe^{-2x}}{-4+4} = \frac{11xe^{-2x}}{0} \quad (\text{failure})$$

$$P.I = \frac{11x^2e^{-2x}}{2}$$

$$\therefore y = C.F + P.I$$

$$\boxed{y = (Ax+B)e^{-2x} + \frac{11x^2e^{-2x}}{2}}$$

3) Solve : $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 4$

Soln:

$$(D^2+4D+3)y = 4$$

The A.E is,

$$m^2+4m+3=0$$

$$m = -1, -3$$

$$C.F = Ae^{-x} + Be^{-3x}$$

$$P.I = \frac{4}{D^2+4D+3}$$

$e^{0x} = e^{ax}$

$a=0$

$D \to a \to 0$

$$= \frac{4 \cdot e^{0x}}{D^2+4D+3} = \frac{4e^{0x}}{3}$$

$$P.I = 4/3$$

$$y = C.F + P.I$$

$$\boxed{y = Ae^{-x} + Be^{-3x} + 4/3}$$