UNIT – IV

FUNCTIONS OF SEVERAL VARIABLES

PARTIAL DERIVATIVES

(1) Find the 1st order and 2nd order partial derivatives of $z = x^3 + y^3 - 3axy$.

Soln:

$$\frac{\partial z}{\partial x} = z_x = 3x^2 - 3ay$$

$$\frac{\partial z}{\partial y} = z_y = 3y^2 - 3ax$$

$$\frac{\partial^2 z}{\partial x^2} = z_{xx} = 6x$$

$$\frac{\partial^2 z}{\partial y^2} = z_{yy} = 6y$$

$$\frac{\partial^2 z}{\partial x \partial y} = z_{xy} = -3a$$

(2) Find the first order partial derivatives of $u = xe^y + ye^x$.

Soln:

$$\frac{\partial u}{\partial x} = e^y + ye^x$$

$$\frac{\partial u}{\partial y} = xe^y + e^x$$

(3) If $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$ then show that $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}$.

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Soln:

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 \longrightarrow (1)$$

Diff (1) w.r.t x partially,

$$2r \frac{\partial r}{\partial x} = 2(x-a)$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x-a}{r}$$

|||ly $\frac{\partial r}{\partial y} = \frac{y-b}{r}$

$$\frac{\partial r}{\partial z} = \frac{z-c}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x-a}{r}\right)$$

$$= \frac{r(1) - (x-a)\frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{1}{r} - \frac{(x-a)}{r^2}\frac{\partial r}{\partial x}$$

$$\therefore \frac{\partial^2 r}{\partial x^2} = \frac{1}{r} - \frac{(x-a)^2}{r^3} \longrightarrow (2)$$

|||ly $\frac{\partial^2 r}{\partial y^2} = \frac{1}{r} - \frac{(y-b)^2}{r^3} \longrightarrow (3)$

$$\frac{\partial^2 r}{\partial z^2} = \frac{1}{r} - \frac{(z-c)^2}{r^3} \longrightarrow (4)$$

(2) + (3) + (4) $\Rightarrow$

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{3}{r} - \frac{1}{r^3}\left[(x-a)^2 + (y-b)^2 + (z-c)^2\right]$$

$$= \frac{3}{r} - \frac{1}{r^3} \cdot r^3$$

$$= \frac{2}{r}$$

4) If $z = (x^2 + xy + y^2)^r$ then show that

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) = 4r^2 z.$$

Soln:

$$z = (x^2 + xy + y^2)^r$$

$$\frac{\partial z}{\partial x} = r(x^2 + xy + y^2)^{r-1}(2x+y)$$

$$\frac{\partial z}{\partial y} = r(x^2 + xy + y^2)^{r-1}(2y+x)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = r(x^2+xy+y^2)^{r-1}[2x^2+2y^2+2xy]$$

$$= 2r(x^2+xy+y^2)^r$$

$$= 2rz$$

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)(2rz)$$

$$= (2r)(2rz)$$

$$= 4r^2 z.$$

5) If $z = f(x+ct) + g(x-ct)$ then prove that

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

Soln:

$$\frac{\partial z}{\partial t} = f'(x+ct)(c) + g'(x-ct)(-c)$$

$$\frac{\partial^2 z}{\partial t^2} = c^2[f''(x+ct) + g''(x-ct)]$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + g''(x-ct)$$

$$\therefore \frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$$

(6) We can prove the following Statements :

(i) $u_{xy} = u_{yx}$ when $u = \tan^{-1}\left(\frac{x}{y}\right)$

(ii) $u = (x^2+y^2+z^2)^{-1/2}$, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

(iii) $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{y}{x}\right)$, $\frac{\partial^2 u}{\partial x \, \partial y} = \frac{x^2-y^2}{x^2+y^2}$

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