TOTAL DERIVATIVES

★ If $u = f(x,y)$ then the total differential of u is given by,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

★ If $u = f(x,y)$ where $x = f(t)$, $y = g(t)$ then the total derivative w.r.t t is given by,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

★ If $u = f(x,y)$ and $y = f(x)$, then the total derivative of u w.r.t x is given by,

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

★ Let $f(x,y) = c$ (c is a constant or zero) be an implicit fn of x and y, then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

① If $u = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$ ST

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

PROBLEMS:

① Find the total differential of $u = \sin(xy^2)$

Soln:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$= y^2\cos(xy^2)\,dx + 2xy\cos(xy^2)\,dy$$

② Find the total derivative of $u = xy\log xy$

Soln:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad \frac{\partial u}{\partial x} = y\left[x\frac{1}{xy}\right.$$

$$= y(1+\log xy)\,dx + x(1+\log xy)\,dy$$

(3) Find $\frac{du}{dt}$ when $u = x^2+y^2+z^2$, $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$.

Soln:

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = 2y \qquad \frac{\partial u}{\partial z} = 2z$$

$$\frac{dx}{dt} = 2e^{2t} \qquad \frac{dy}{dt} = e^{2t}(-3\sin 3t) + 2e^{2t}\cos 3t \qquad \frac{dz}{dt} = e^{2t}(3\cos \ldots + 2e^{2t}\sin \ldots$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= 2x(2e^{2t}) + 2ye^{2t}(2\cos 3t - 3\sin 3t) + 2ze^{2t}(3\cos 3t + 2\sin 3t)$$

Replace x, y, z interms of t

$$\frac{du}{dt} = 4e^{2t} + 2e^{4t}\cos 3t(2\cos 3t - 3\sin 3t) + 2e^{2t}\sin 3t(3\cos 3t + 2\sin 3t)$$

(4) Find $\frac{du}{dt}$ (i) when $u = x^2+y^2$, $x = at^2$, $y = 2at$ (ii) $u = \sin\left(\frac{x}{y}\right)$, $x = e^t$, $y = t^2$ (iii) $u = xy + yz + z\ldots$ $x = e^t$, $y = e^{-t}$, $z = 1/t$.

Soln:

(i) $$\frac{du}{dt} = 2x(2at) + 2y(2a)$$

$$= 4a^2t^3 + 8a^2t$$

(ii) $$\frac{du}{dt} = \frac{1}{y}\cos\left(\frac{x}{y}\right)e^t - \frac{x}{y^2}\cos\left(\frac{x}{y}\right)2t$$

$$= \left(1 - \frac{2}{t^2}\right)\left(\frac{e^t}{t^2}\right)\cos\left(\frac{e^t}{t^2}\right)$$

(iii) $\frac{du}{dt} = (y+z)e^{t} + (x+z)(-e^{-t}) + (x+y)\left(\frac{-1}{t^2}\right)$

$= \frac{1}{t} 2 \sin ht - \frac{1}{t^2} 2 \cos h t.$

5) Find $\frac{dy}{dx}$ if $x^3+y^3 = 3ax^2y$

Soln:

$f(x,y) = x^3+y^3-3ax^2y$

$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$

$= \frac{-(3x^2-6axy)}{3y^2-3ax^2}$

$= \frac{2axy-x^2}{y^2-ax^2}$

or

$3x^2+3y^2\frac{dy}{dx} = 3a\left(x^2\frac{dy}{dx}+y2x\right)$

$3y^2\frac{dy}{dx} - 3ax^2\frac{dy}{dx} = 6axy - 3x^2$

$\frac{dy}{dx} = \frac{2axy-x^2}{y^2-ax^2}$

$\frac{d}{dx}(a^x) = a^x \log a$

6) Find $\frac{dy}{dx}$ if $x^y + y^x = c.$

Soln:

$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y} = \frac{-yx^{y-1}+y^x \log y}{x^y \log x + xy^{x-1}}$

7) Find $\frac{du}{dx}$ if $u = \sin(x^2+y^2)$ where $a^2x^2+b^2y^2=c^2$

Soln:

$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$

$= 2x\cos(x^2+y^2) + 2y\cos(x^2+y^2)\left(\frac{-a^2x}{b^2y}\right)$

$= 2\cos(x^2+y^2)\left(x - \frac{a^2x}{b^2}\right)$

$a^2x^2+b^2y^2=c^2$

$2b^2y\frac{dy}{dx} = -2a^2x$

$\frac{dy}{dx} = \frac{-a^2x}{b^2y}$

$b^2y^2 = c^2-a^2x^2$

$b^2 \cdot 2y\frac{dy}{dx} = 0 - a^2(2x) \Rightarrow \frac{dy}{dx} = \frac{-a^2x}{b^2y}$