Method of Lagrangian Multipliers

Working rule :

* Let $f(x,y,z)$ be the function of three variables subject to the constraint $\varphi(x,y,z)=0$
* Set the auxiliary function as,

$$g = f+\lambda\varphi$$

where λ is the Lagrangian multiplier.

* Set $\frac{\partial g}{\partial x}=0$, $\frac{\partial g}{\partial y}=0$, $\frac{\partial g}{\partial z}=0$
* On solving the above equations, we get the values of x, y and z. Substituting x, y & z in φ we get value of λ. They are called stationary values.
* On substitution of x, y and z in the given function we get the required extreme values.

Problems

(1) Find the minimum value of $x^2+y^2+z^2$ subject to the condition $x+y+z=3a$

Soln :

$$f(x, y, z) = x^2 + y^2 + z^2$$

$$\varphi(x, y, z) = x + y + z - 3a$$

Let the auxiliary function be,

$$g = f + \lambda\varphi$$

$$g = x^2 + y^2 + z^2 + \lambda(x + y + z - 3a)$$

$$\frac{\partial g}{\partial x} = 0 \Rightarrow 2x + \lambda = 0 \Rightarrow 2x = -\lambda \rightarrow (1)$$

$$\frac{\partial g}{\partial y} = 0 \Rightarrow 2y + \lambda = 0 \Rightarrow 2y = -\lambda \rightarrow (2)$$

$$\frac{\partial g}{\partial z} = 0 \Rightarrow 2z + \lambda = 0 \Rightarrow 2z = -\lambda \rightarrow (3)$$

From (1), (2) & (3),

$$2x = 2y = 2z$$

$$\Rightarrow x = y = z \rightarrow (4)$$

Given : $x + y + z = 3a$

$x + x + x = 3a$ (using (4))

$3x = 3a$

$\boxed{x = a}$

$\therefore \boxed{x = y = z = a}$

(a, a, a) is the point where minimum value occurs.

$\therefore$ Minimum value $= x^2 + y^2 + z^2 = a^2 + a^2 + a^2$

$= 3a^2$