Evolute :

The locus of the centre of curvature of the given curve is called the evolute of the curve.

Working procedure to find the evolute :

1. Write the parametric form of the given curve.
2. Find centre of curvature $(\bar{x}, \bar{y})$.
3. Eliminate the parameter from $\bar{x}$ and $\bar{y}$
4. Taking locus of the above equation, we get the required evolute.

Curve	Cartesian equation	Parametric equation
Parabola	① $y^2 = 4ax$	① $x = at^2$, $y = 2at$
	② $x^2 = 4ay$	② $x = 2at$, $y = at^2$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a\cos\theta$, $y = b\sin\theta$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x = a\sec\theta$, $y = b\tan\theta$
Rectangular hyperbola	$xy = c^2$	$x = ct$, $y = \frac{c}{t}$
Astroid	$x^{2/3} + y^{2/3} = a^{2/3}$	$x = a\cos^3\theta$, $y = a\sin^3\theta$

Problems :

(1) Find the equation of the evolute of the Parabola $y^2 = 4ax$.

Soln :

The parametric equations of the parabola $y^2 = 4ax$ are $x = at^2$, $y = 2at$.

$$\frac{dx}{dt} = 2at \quad , \quad \frac{dy}{dt} = 2a .$$

$$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{t}\right)$$

$$= \frac{d}{dx}(t^{-1})$$

$$= \frac{-1}{t^2} \cdot \frac{dt}{dx}$$

$$= \frac{-1}{t^2} \cdot \frac{1}{dx/dt}$$

$$= \frac{-1}{t^2} \cdot \frac{1}{2at}$$

$$y_2 = \frac{-1}{2at^3}$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$= at^2 - \frac{1/t}{-1/2at^3}\left(1+\frac{1}{t^2}\right)$$

$$= at^2 + \frac{1}{\cancel{t}}\, 2at^{\cancel{3}}\left(1+\frac{1}{t^2}\right)$$

$$= at^2 + 2at^2\left(1+\frac{1}{t^2}\right)$$

$$= at^2 + 2at^2 + 2a\cancel{t^2}\cdot\frac{1}{\cancel{t^2}}$$

$$= at^2 + 2at^2 + 2a$$

$$\bar{x} = 3at^2 + 2a \Rightarrow t^2 = \frac{\bar{x}-2a}{3a} \rightarrow (1)$$

$$\bar{y} = y + \frac{1}{y_2}(1+y_1^2)$$

$$= 2at + \frac{1}{-1/2at^3}\left(1+\frac{1}{t^2}\right)$$

$$= 2at - 2at^3\left(1+\frac{1}{t^2}\right)$$

$$= 2at - 2at^3 - 2at^{\cancel{3}}\cdot\frac{1}{\cancel{t^2}}$$

$$= \cancel{2at} - 2at^3 - \cancel{2at}$$

$$\bar{y} = -2at^3 \Rightarrow t^3 = -\frac{\bar{y}}{2a} \rightarrow (2)$$

Taking cube of ① & Squaring ② we get,

$$(t^2)^3 = \left(\frac{\bar{x} - 2a}{3a}\right)^3$$

$$t^6 = \frac{(\bar{x} - 2a)^3}{27a^3} \longrightarrow ③$$

$$(t^3)^2 = \left(\frac{-\bar{y}}{2a}\right)^2$$

$$t^6 = \frac{\bar{y}^2}{4a^2} \longrightarrow ④$$

From ③ & ④,

$$\frac{(\bar{x} - 2a)^3}{27a^{\cancel{3}}} = \frac{\bar{y}^2}{4a^{\cancel{2}}}$$

$$\frac{(\bar{x} - 2a)^3}{27a} = \frac{\bar{y}^2}{4}$$

$$4(\bar{x} - 2a)^3 = 27a\bar{y}^2$$

Locus of $(\bar{x}, \bar{y})$ is,

$$\boxed{4(x - 2a)^3 = 27ay^2}$$

which is the evolute of the parabola $y^2 = 4ax$.