

## Centre of Curvature

$$\bar{x} = x - \frac{y_1}{y_2} (1 + y_1^2)$$

$$\bar{y} = y + \frac{1}{y_2} (1 + y_1^2)$$

$(\bar{x}, \bar{y})$  is the coordinate of the centre of curvature.

## Circle of Curvature

The equation of the circle of curvature is

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$$

## Problems

- ① Find the centre of curvature of  $y = x^2$  at the origin.

Soln:

$$y = x^2$$

Point:  $(0, 0)$

$$\frac{dy}{dx} = y_1 = 2x \quad ; \quad y_1(0, 0) = 2(0) = 0$$

$$\frac{d^2y}{dx^2} = y_2 = 2 \quad ; \quad y_2(0, 0) = 2$$

The centre of curvature is,

$$\bar{x} = x - \frac{y_1}{y_2} (1 + y_1^2)$$

$$= 0 - 0(1 + 0)$$

$$\bar{x} = 0$$

$$\bar{y} = y + \frac{1}{y_2} (1 + y_1^2)$$

$$= 0 + \frac{1}{2} (1 + 0)$$

$$\bar{y} = \frac{1}{2}$$

$$\therefore (\bar{x}, \bar{y}) = (0, \frac{1}{2})$$

- ② Find the equation of the circle of curvature of the rectangular hyperbola  $xy = 12$  at the point  $(3, 4)$ .

Soln:

The equation of circle of curvature is given by,

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2 \rightarrow \textcircled{1}$$

where,

$$\bar{x} = x - \frac{y_1 (1 + y_1^2)}{y_2}$$

$$\bar{y} = y + \frac{1 (1 + y_1^2)}{y_2}$$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

Given:  $xy = 12$  point:  $(3, 4)$

$$x \cdot \frac{dy}{dx} + y(1) = 0$$

$$x y_1 + y = 0$$

$$x y_1 = -y$$

$$y_1 = \frac{-y}{x}$$

$$y_1(3, 4) = \frac{-4}{3}$$

$$y_2 = \frac{x(-y_1) - (-y)(1)}{x^2}$$

$$= \frac{-x y_1 + y}{x^2}$$

$$y_2(3,4) = \frac{-\cancel{3} \left( \frac{-4}{\cancel{3}} \right) + 4}{3^2}$$

$$= \frac{4+4}{9}$$

$$\boxed{y_2 = \frac{8}{9}}$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= 3 - \frac{\left( \frac{-4}{3} \right) \left( 1 + \left( \frac{-4}{3} \right)^2 \right)}{\frac{8}{9}}$$

$$= 3 + \frac{4}{3} \times \frac{9}{8} \left( 1 + \frac{16}{9} \right)$$

$$= 3 + \frac{3}{2} \left( \frac{9+16}{9} \right)$$

$$= 3 + \frac{3}{2} \times \frac{25}{3}$$

$$= 3 + \frac{25}{6}$$

$$= \frac{18+25}{6}$$

$$\boxed{\bar{x} = \frac{43}{6}}$$

$$\bar{y} = y + \frac{1}{y_2} (1 + y_1^2)$$

$$= 4 + \frac{1}{\frac{8}{9}} \left[ 1 + \left( \frac{-4}{3} \right)^2 \right]$$

$$= 4 + \frac{9}{8} \left[ 1 + \frac{16}{9} \right]$$

$$= 4 + \frac{9}{8} \left( \frac{9+16}{9} \right)$$

$$= 4 + \frac{9}{8} \times \frac{25}{9} = 4 + \frac{25}{8} = \frac{32+25}{8}$$

$$\boxed{\bar{y} = \frac{57}{8}}$$

$$p = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

$$= \frac{\left[ 1 + \left( \frac{-4}{3} \right)^2 \right]^{3/2}}{8/9}$$

$$= \frac{\left[ 1 + \frac{16}{9} \right]^{3/2}}{8/9} = \frac{\left( \frac{25}{9} \right)^{3/2}}{8/9}$$

$$= \frac{\left[ \frac{5^2}{3^2} \right]^{3/2}}{\frac{8}{9}} = \frac{5^3}{3^3} \times \frac{9}{8}$$

$$= \frac{125}{27} \times \frac{9}{8} = \frac{125}{24} \Rightarrow \boxed{p = \frac{125}{24}}$$

$$\therefore \textcircled{1} \Rightarrow \left(x - \frac{43}{6}\right)^2 + \left(y - \frac{57}{8}\right)^2 = \left(\frac{125}{24}\right)^2$$