Taylor's expansion

Let $f(x,y)$ be a function of two variables x and y.

The Taylor Series expansion about the point (a,b) is given by,

$$f(x,y) = f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b) + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

$$+ \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) \cdot f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right]$$

$$+ \cdots$$

Problems:

1) Expand $x^2 y + 3y - 2$ in powers of $x-1$ and $y+2$ using Taylor's expansion.

Solution:

Given: $f(x,y) = x^2 y + 3y - 2$

$x - 1 = 0$

$x = 1$

$y + 2 = 0$

$y = -2$

$\therefore$ $a = 1$

$b = -2$

$f(x,y) = x^2y+3y-2$	At $(a,b) = (1,-2)$
$f(x,y) = x^2y+3y-2$	$f(1,-2) = 1(-2)+3(-2)-2$ $= -10$
$f_x = 2xy$	$f_x(1,-2) = 2(1)(-2) = -4$
$f_{xx} = 2y$	$f_{xx}(1,-2) = 2(-2) = -4$
$f_{xxx} = 0$	$f_{xxx}(1,-2) = 0$
$f_y = x^2+3$	$f_y(1,-2) = 1+3 = 4$
$f_{yy} = 0$	$f_{yy}(1,-2) = 0$
$f_{yyy} = 0$	$f_{yyy}(1,-2) = 0$
$f_{xy} = 2x$	$f_{xy}(1,-2) = 2(1) = 2$
$f_{xxy} = 2$	$f_{xxy}(\cancel{1},-2) = 2$
$f_{xyy} = 0$	$f_{xyy}(1,-2) = 0$

Taylor's Series expansion is,

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b)$$

$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

$$+\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right]$$

$$+ \cdots$$

$$f(x,y) = f(1,-2) + (x-1)f_x(1,-2) + (y+2)f_y(1,-2)$$

$$+\frac{1}{2!}\left[(x-1)^2 f_{xx} + 2(x-1)(y+2)f_{xy} + (y+2)^2 f_{yy}\right]$$

$$+\frac{1}{3!}\left[(x-1)^3 f_{xxx} + 3(x-1)^2(y+2) f_{xxy} + 3(x-1)(y+2)^2 f_{xyy} + (y+2)^3 f_{yyy}\right] + \cdots$$

$$\therefore f(x,y) = -10 + (x-1)(-4) + (y+2)4 + \frac{1}{2}\left[(x-1)^2(-4) + 2(x-1)(y+2)2 + 0\right] + \frac{1}{6}\left[0 + 3(x-1)^2(y+2)2 + 0 + 0\right]$$

$$f(x,y) = -10 - 4(x-1) + 4(y+2) + \frac{1}{\cancel{2}}\left[-\cancel{4}(x-1)^2 + \cancel{4}(x-1)(y+2)\right] + \frac{1}{\cancel{6}}\left[\cancel{6}(x-1)^2(y+2)\right] + \cdots$$

$$f(x,y) = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2) + \cdots$$