Envelopes :

A curve which touches each member of a family of curve is called the envelope of the family of curves.

Problems :

(1) Find the envelope of the family of straight lines :

(i) $y = mx + am^2$

(ii) $y = mx + \frac{1}{m}$

(iii) $y = mx + \frac{a}{m}$

(iv) $y = mx + \sqrt{a^2m^2 + b^2}$

(v) $y = mx + \sqrt{1+m^2}$

where m is the parameter

(vi) $x\cos\alpha + y\sin\alpha = a\sec\alpha$, where α is the parameter.

Soln :

(i) $y = mx + am^2$

$$mx + am^2 - y = 0$$

$$am^2 + mx - y = 0$$

which is quadratic in 'm'.

Envelope is $B^2 - 4AC = 0$

Here $A = a$, $B = x$, $C = -y$

$\therefore B^2 - 4AC = 0 \Rightarrow x^2 - 4a(-y) = 0$

$$\boxed{x^2 + 4ay = 0}$$

(ii) $y = mx + \frac{1}{m}$

$$y = \frac{m^2x + 1}{m}$$

$$my = m^2x + 1$$

$$m^2x + 1 - my = 0$$

$m^2x - my + 1 = 0$ which is quadratic in 'm'.

$A = x$, $B = -y$, $C = 1$

Envelope is $B^2 - 4AC = 0$

$$(-y)^2 - 4(x)(1) = 0$$

$$y^2 - 4x = 0$$

$$\boxed{y^2 = 4x}$$

(iii) $y = mx + \frac{a}{m}$

$$y = \frac{m^2x + a}{m}$$

$$my = m^2x + a$$

$$m^2x + a - my = 0$$

$m^2 x - my + a = 0$ which is quadratic in m.

$A = x, \; B = -y, \; C = a$

$B^2 - 4AC = 0$

$(-y)^2 - 4(x)(a) = 0$

$$\boxed{y^2 - 4ax = 0}$$

(iv) $y = mx + \sqrt{a^2m^2 + b^2}$

$y - mx = \sqrt{a^2m^2+b^2}$

Squaring on both sides,

$(y-mx)^2 = a^2m^2 + b^2$

$y^2 + m^2x^2 - 2ymx = a^2m^2 + b^2$

$y^2 + m^2x^2 - 2ymx - a^2m^2 - b^2 = 0$

$m^2(x^2 - a^2) - 2xym + (y^2 - b^2) = 0$

which is quadratic in m.

$A = x^2 - a^2, \; B = -2xy, \; C = y^2 - b^2$

$B^2 - 4AC = 0$

$\Rightarrow (-2xy)^2 - 4(x^2-a^2)(y^2-b^2) = 0$

$4x^2y^2 - 4(x^2-a^2)(y^2-b^2) = 0$

$\div$ by 4 $\Rightarrow x^2y^2 - (x^2-a^2)(y^2-b^2) = 0$

$x^2y^2 - (x^2y^2 - x^2b^2 - a^2y^2 + a^2b^2) = 0$

$\cancel{x^2y^2} - \cancel{x^2y^2} + x^2b^2 + a^2y^2 - a^2b^2 = 0$

$x^2b^2 + a^2y^2 = a^2b^2$

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$\div$ by $a^2b^2 \Rightarrow \dfrac{x^2b^2}{a^2b^2} + \dfrac{a^2y^2}{a^2b^2} = \dfrac{a^2b^2}{a^2b^2}$

$$\Rightarrow \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

(v) $y = mx + \sqrt{1+m^2}$

$y - mx = \sqrt{1+m^2}$

Squaring on both sides,

$(y-mx)^2 = 1+m^2$

$y^2 + m^2x^2 - 2ymx = 1+m^2$

$y^2 + m^2x^2 - 2ymx - 1 - m^2 = 0$

$m^2(x^2-1) - 2xym + (y^2-1) = 0$

which is quadratic in 'm'.

$A = x^2-1$, $B = -2xy$, $C = y^2-1$

$B^2 - 4AC = 0 \Rightarrow (-2xy)^2 - 4(x^2-1)(y^2-1) = 0$

$4x^2y^2 - 4(x^2-1)(y^2-1) = 0$

$\div 4 \Rightarrow \quad x^2y^2 - (x^2-1)(y^2-1) = 0$

$x^2y^2 - (x^2y^2 - x^2 - y^2 + 1) = 0$

$\cancel{x^2y^2} - \cancel{x^2y^2} + x^2 + y^2 - 1 = 0$

$x^2 + y^2 - 1 = 0$

$$\boxed{x^2 + y^2 = 1}$$

(vi) $x\cos\alpha + y\sin\alpha = a\sec\alpha$

$\div$ by $\cos\alpha$,

$$\frac{x\cos\alpha}{\cos\alpha} + \frac{y\sin\alpha}{\cos\alpha} = \frac{a\sec\alpha}{\cos\alpha}$$

$$x + y\tan\alpha = a\sec^2\alpha$$

$$x + y\tan\alpha = a(1+\tan^2\alpha)$$

$$x + y\tan\alpha = a + a\tan^2\alpha$$

$$a + a\tan^2\alpha - x - y\tan\alpha = 0.$$

$$a\tan^2\alpha - y\tan\alpha + a - x = 0$$

Which is quadratic in 'α'.

$A = a$, $B = -y$, $C = a - x$

$B^2 - 4AC = 0$.

$(-y)^2 - 4a(a-x) = 0$

$$\boxed{y^2 = 4a(a-x)}$$