JACOBIANS

If $u = f(x,y)$, $v = g(x,y)$ are two functions then Jacobian of u and v w.r.t x, y is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ (or) J and is defined by,

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

For three variables (u, v, w) which are functions of (x, y, z) then Jacobian is,

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

PROPERTIES OF JACOBIANS :

* If u and v are functions of x and y and x and y are functions of r and s, then

$$\frac{\partial(u,v)}{\partial(r,s)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,s)}$$

* u and v are functions of x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1 \Rightarrow JJ' = 1$$

* Functional dependence

If $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$, then u, v and w are functionally dependent.

PROBLEMS :

1. Find the Jacobian of the following transformati

(i) $u = \frac{2x-y}{2}$, $v = y/2$. Find $\frac{\partial(u,v)}{\partial(x,y)}$

(ii) $u = xyz$, $v = xy+yz+zx$, $w = x+y+z$. Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

(iii) $u = 2xy$, $v = x^2/y$

(iv) $u = y^2/x$, $v = x^2/y$

Find $\frac{\partial(u,v)}{\partial(x,y)}$

(V) $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ Find $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$

(vi) $x = u(1+v)$, $y = v(1+u)$. Find $\frac{\partial(x,y)}{\partial(u,v)}$

(vii) $x = \frac{u^2}{v}$, $y = \frac{v^2}{w}$, $z = \frac{w^2}{u}$. Find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

(viii) $u = e^x \cos y$, $v = e^x \sin y$. Find $\frac{\partial(u,v)}{\partial(x,y)}$

(ix) $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$. Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Soln:

(i) $J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & -1/2 \\ 0 & 1/2 \end{vmatrix} = 1/2$

(ii) $J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \frac{\partial(u,v,w)}{\partial(x,y,z)}$

$= \begin{vmatrix} yz & xz & xy \\ y+z & x+z & x+y \\ 1 & 1 & 1 \end{vmatrix}$

$= (x-y)(y-z)(z-x)$

(iii) $J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$= \begin{vmatrix} 2y & 2x \\ \frac{2x}{y} & \frac{-x^2}{y^2} \end{vmatrix}$

$= \frac{-6x^2}{y}$

(iv) $J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -y^2/x^2 & 2y/x \\ 2x/y & -x^2/y^2 \end{vmatrix} = -3$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{J} = -1/3$$

(v) $J = \dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r$$

(vi) $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$

$$= 1+u+v$$

(vii) $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{2u}{v} & \frac{-u^2}{v^2} & 0 \\ 0 & \frac{2v}{w} & \frac{-v^2}{w^2} \\ \frac{-w^2}{u^2} & 0 & \frac{2w}{u} \end{vmatrix}$$

$$= 7$$

(viii) $J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix}$

$= e^{2x}$

(ix) $J = \begin{vmatrix} \dfrac{-yz}{x^2} & \dfrac{z}{x} & \dfrac{y}{x} \\ z/y & -zx/y^2 & \dfrac{x}{y} \\ y/z & x/z & -xy/z^2 \end{vmatrix} = 4$

② If $u = 2xy$, $v = x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$

Evaluate $\dfrac{\partial(r,\theta)}{\partial(u,v)}$.

Soln:

u, v

x, y

r, θ

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= -4r^3$$

$$\frac{\partial(r,\theta)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(r,\theta)}} = \frac{-1}{4r^3}$$

③ If $x+y+z = u$, $y+z = uv$, $z = uvw$ find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

Soln:

$z = uvw$, $y = uv - z$ $\quad x = u - y - z$

$= uv - uvw \quad = u - uv + uvw - uvw$

$= u - uv$

$\therefore x = u - uv$, $y = uv - uvw$, $z = uvw$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix} = u^2 v$$

④ If $y_1 = 1 - x_1$, $y_2 = x_1(1-x_2)$, $y_3 = x_1 x_2 (1-x_3)$

Find $\dfrac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)}$

Soln:

$$J = \frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 0 \\ 1-x_2 & -x_1 & 0 \\ x_2 - x_2 x_3 & -x_1 x_3 & -x_1 x_2 \end{vmatrix}$$

$$= -x_1^2 x_2.$$

⑤ If $y_1 = \dfrac{x_2 x_3}{x_1}$, $y_2 = \dfrac{x_1 x_3}{x_2}$, $y_3 = \dfrac{x_1 x_2}{x_3}$

Show that $J = 4$.

Soln: y_1, y_2, y_3 w.r.t x_1, x_2, x_3 if $y_1 = \dfrac{x_2 x_3}{x_1}$, $y_2 = x_3 x_1 / x_2$ and $y_3 = \dfrac{x_1 x_2}{x_3}$

$$J = \begin{vmatrix} \dfrac{-x_2 x_3}{x_1^2} & \dfrac{x_3}{x_1} & \dfrac{x_2}{x_1} \\ \dfrac{x_3}{x_2} & \dfrac{-x_1 x_3}{x_2^2} & \dfrac{x_1}{x_2} \\ \dfrac{x_2}{x_3} & \dfrac{x_1}{x_3} & \dfrac{-x_1 x_2}{x_3^2} \end{vmatrix} = 4$$

(6) If $u = \dfrac{x+y}{x-y}$, $v = \tan^{-1}x + \tan^{-1}y$. Find the Jacobian.

Soln:

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{-2y}{(x-y)^2} & \dfrac{2x}{(x-y)^2} \\ \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{-2}{(x-y)^2}\left[\frac{y}{1+y^2} + \frac{x}{a+x^2}\right]$$

(7) Show that the function $u = \dfrac{x}{y}$, $v = \dfrac{x+y}{x-y}$ are functionally dependent and also find their relation between them.

Soln:

$$J = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{y} & \dfrac{-x}{y^2} \\ \dfrac{-2y}{(x-y)^2} & \dfrac{2x}{(x-y)^2} \end{vmatrix} = 0$$

$\therefore$ u and v are functionally dependent.

$$v = \frac{x+y}{x-y} = \frac{x}{x}\left[\frac{1+y/x}{1-y/x}\right]$$

$v = \dfrac{1+u}{1-u}$ is the relation b/w them.

(8) Show that $u = 2x - y + 3z$, $v = 2x - y - z$, $w = 2x - y + z$ are functionally dependent. Find their relation.

Soln:

$$J = \begin{vmatrix} 2 & -1 & 3 \\ 2 & -1 & -1 \\ 2 & -1 & 1 \end{vmatrix} = 0$$

$\therefore$ They are functionally dependent.

$2x - y = -3z + u \qquad 2x - y = z + v \qquad 2x - y = w - z$

$\Rightarrow u - 3z = v + z = w - z$

$\Rightarrow u - 3z = w - z \Rightarrow u - w - 2z = 0$

$v + z = w - z \Rightarrow \underline{v - w + 2z = 0}$

$u + v - 2w = 0$.

(9) If $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$, $w = x + y + z$ then ST u, v, w are not independent of one another. Also find the relationship among them

Soln:

$J = 0$

$\therefore$ Functionally dependent or u, v and w are independent of one another.

$w = x + y + z$

$w^2 = (x + y + z)^2$

$= x^2 + y^2 + z^2 + 2(xy + yz + zx)$

$w^2 = v + 2u$

$2u + v - w^2 = 0$