UNIT - III

DIFFERENTIAL CALCULUS

I. Basic differentiation formulas :

S.No	y	dy/dx
1	K (constant)	0
2	x^n	$n x^{n-1}$
3	e^{ax}	$a e^{ax}$
4	$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
5	$\log x$	$\frac{1}{x}$
6	a^x	$a^x \log a$
7	$\sin x$	$\cos x$
8	$\cos x$	$-\sin x$
9	$\tan x$	$\sec^2 x$
10	$\cot x$	$-\operatorname{cosec}^2 x$
11	$\sec x$	$\sec x \tan x$
12	$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
13	$\sinh x$	$\cosh x$
14	$\cosh x$	$\sinh x$

1. $\frac{d}{dx}(\text{constant}) = 0$

Ex: $\frac{d}{dx}(1) = 0$, $\frac{d}{dx}(k) = 0$, ...

2. $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\frac{d}{dx}(x^2) = 2x^{2-1} = 2x$$

$$\frac{d}{dx}(x^4) = 4x^{4-1} = 4x^3$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)\,x^{(-1-1)} = (-1)\,x^{-2}$$
$$= \frac{-1}{x^2}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

3. $\frac{d}{dx}(e^{ax}) = ae^{ax}$

$$\frac{d}{dx}(e^{2x}) = 2e^{2x}$$

$$\frac{d}{dx}(e^{-x}) = (-1)e^{-x} = -e^{-x}$$

4. $\frac{d}{dx}(\log x) = \frac{1}{x}$

$$\frac{d}{dx}(\log x^2) = \frac{d}{dx}(2\log x) = 2\frac{d}{dx}(\log x) = \frac{2}{x}$$

II. <u>Product rule</u> :

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} = uv' + vu'$$

where u & v are functions of x.

III. <u>Quotient rule</u> :

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{vu' - uv'}{v^2}$$

IV. If $x = x(t)$, $y = y(t)$ where 't' is a Parameter then,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dt}{dt}$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{1}{dx/dt}$$

Curvature of a curve :

The rate of bending of a curve in any interval is called the curvature of the curve in that interval. It is denoted by K.

Note : The curvature of a straight line is zero.

Radius of curvature :

It is defined as the reciprocal of the curvature of the curve and is denoted by ρ.

$$\boxed{\rho = \frac{1}{K}}$$

Note :

The radius of curvature at every point of the circle is equal to the radius of the circle. i.e, $\rho = r$

Hence the curvature is, $K = \frac{1}{\rho} = \frac{1}{r}$

Formula for radius of curvature :

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} \quad \text{(or)} \quad \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

where $y_1 = \frac{dy}{dx}$, $y_2 = \frac{d^2y}{dx^2}$

Note : When $\frac{dy}{dx}$ becomes ∞, $\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{d^2x/dy^2}$

is the alternative formula for radius of curvature.

Problems :

(1) Find the radius of curvature at any point of the Catenary $y = c\cosh(x/c)$.

Soln :

$$y = c\cosh(x/c) \longrightarrow (1)$$

$$y_1 = c \cdot \sinh(x/c) \cdot \frac{1}{c} = \sinh(x/c)$$

$$y_2 = \cosh\left(\frac{x}{c}\right) \cdot \frac{1}{c}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{\left(1+\sinh^2(x/c)\right)^{3/2}}{\cosh(x/c)} \cdot c$$

$\cosh^2 x - \sinh^2 x = 1$

$$= \frac{\left(\cosh^2(x/c)\right)^{3/2}}{\cosh(x/c)} \cdot c$$

$$\rho = c\cosh^2(x/c) \longrightarrow (2)$$

(2) Find ρ for (i) $y = \log \sin x$ (ii) $y = c \log \sec (x/c)$

Soln :

(i) $y = \log \sin x$

$$y_1 = \frac{1}{\sin x} \cos x = \cot x$$

$$y_2 = -\operatorname{cosec}^2 x$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\cot^2 x)^{3/2}}{-\operatorname{cosec}^2 x}$$

$$= \frac{\operatorname{cosec}^3 x}{-\operatorname{cosec}^2 x} = -\operatorname{cosec} x$$

$$\boxed{|\rho| = \operatorname{cosec} x}$$

(ii) $y = c \log \sec (x/c)$

$$y_1 = \cancel{c} \frac{1}{\sec (x/c)} \cancel{\sec}\left(\frac{x}{c}\right) \tan\left(\frac{x}{c}\right) \cdot \frac{1}{\cancel{c}}$$

$$y_1 = \tan (x/c)$$

$$y_2 = \frac{1}{c} \sec^2 \left(\frac{x}{c}\right)$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\tan^2 (x/c)\right)^{3/2}}{\frac{1}{c} \sec^2 (x/c)}$$

$$\rho = c \frac{\sec^3 (x/c)}{\sec^2 (x/c)} \Rightarrow \boxed{\rho = c \sec (x/c)}$$

(3) Find the radius of curvature for the curves

(i) $y = e^x$, where it crosses the y-axis

(ii) $y^2 = x^3 + 8$ at the point $(-2, 0)$

(iii) $xy = c^2$ at $x = c$

(iv) $y^2 = \frac{a^3 - x^3}{x}$ at $(a, 0)$

(V) $y = \frac{\log x}{x}$ at $x = 1$

(vi) $x^3 + y^3 = 3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Soln:

(i) $y = e^x$

On y-axis $x = 0$

when $x = 0$, $y = e^0 = 1$ $\therefore y = 1$

$\therefore$ The point is $(0, 1)$

$y = e^x$

$y_1 = e^x$; y_1 at $(0,1) = 1$

$y_2 = e^x$; y_2 at $(0,1) = 1$

$$\therefore \rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{1} = 2^{3/2}$$

$$\boxed{\rho = 2\sqrt{2}}$$

(ii) $y^2 = x^3 + 8$

Diff w.r.t 'x', we get

$$2yy_1 = 3x^2$$

$$y_1 = \frac{3x^2}{2y} \quad ; \quad y_1(-2,0) = \infty$$

$$\therefore \frac{dx}{dy} = 0 \quad ; \quad \frac{dx}{dy} = \frac{2y}{3x^2}$$

$$\frac{d^2x}{dy^2} = \frac{(3x^2)2 - (2y)\,6x\,dx/dy}{(9x^4)}$$

$$= \frac{6x^2 - 12xy\,(dx/dy)}{9x^4}$$

$$\left(\frac{d^2x}{dy^2}\right)_{(-2,0)} = \frac{6 \times 4}{9 \times 16} = \frac{1}{6}$$

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{d^2x/dy^2} = \frac{(1+0)^{3/2}}{1/6}$$

$$\boxed{\rho = 6}$$

(iii) $xy = c^2$ at $x = c$

If $x = c$, $xy = c^2$ will be

$$y = c^2/x = c^2/c = c$$

$$y = c$$

$\therefore$ The point is (c, c).

$$xy = c^2$$

$$x y_1 + y = 0$$

$$y_1 = -y/x$$

y_1 at (c, c) is $= -c/c = -1$

$$y_2 = \frac{x(-dy/dx) - (-y).1}{x^2}$$

$$= \frac{-x \times (-y/x) + y}{x^2} = \frac{2y}{x^2}$$

y_2 at $(c, c) = \frac{2c}{c^2} = \frac{2}{c}$

$$\therefore \rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1 + (-1)^2)^{3/2}}{2/c} = \frac{2^{3/2} c}{2}$$

$$= \frac{2\sqrt{2}\, c}{2}$$

$$\boxed{\rho = \sqrt{2} . c}$$

(iv) $y^2 = \dfrac{a^3 - x^3}{x}$ at $(a, 0)$

$$2yy_1 = \frac{-a^3}{x^2} - 2x$$

$$y_1 = \frac{-a^3}{2yx^2} - \frac{x}{y} = \frac{-a^3 - 2x^3}{2x^2 y}$$

y_1 at $(a, 0) = \infty$

$\therefore \dfrac{dx}{dy} = 0$; $\dfrac{dx}{dy} = \dfrac{-2x^2 y}{+a^3 + 2x^3}$

$$\frac{d^2x}{dy^2} = \frac{(a^3 + 2x^3)(-2)\left(x^2 + 2x\frac{dx}{dy} \cdot y\right) + 2x^2 y\left(6x^2 \frac{dx}{dy}\right)}{(a^3 + 2x^3)^2}$$

$$\left(\frac{d^2x}{dy^2}\right)_{(a,0)} = \frac{-2(3a^3)a^2}{9a^6} = \frac{-2}{3a}$$

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{d^2x/dy^2} = \frac{1}{-2/3a}$$

$$\rho = \frac{-3a}{2}$$

$$\boxed{|\rho| = \frac{3a}{2}}$$

(x) $y = \dfrac{\log x}{x}$ at $x = 1$

$$y_1 = \frac{1 - \log x}{x^2}$$

When $x = 1$, $y = 0$

$\therefore (1, 0)$ is the point

$y_1(1, 0) = 1$

$$y_2 = \frac{x^2(-1/x) - (1 - \log x)2x}{x^4}$$

$$= \frac{-x - (1 - \log x)2x}{x^4} = \frac{-1 - (1 - \log x)2}{x^3}$$

$y_2(1, 0) = -1 - 2 = -3$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{(2)^{3/2}}{-3} = \frac{-2\sqrt{2}}{3}$$

$$\boxed{|\rho| = \frac{2\sqrt{2}}{3}}$$

(xi) $x^3 + y^3 = 3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

$$3x^2 + 3y^2 y_1 = 3a(xy_1 + y)$$

$$3y_1(y^2 - ax) = 3(ay - x^2)$$

$$y_1 = \frac{ay - x^2}{y^2 - ax}$$

$$y_1 \text{ at } \left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{a \times \frac{3a}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - a \times \frac{3a}{2}}$$

$$= \frac{-\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)}{\frac{9a^2}{4} - \frac{3a^2}{2}} = -1$$

$$y_2 = \frac{(y^2 - ax)(ay_1 - 2x) - (ay - x^2)(2yy_1 - a)}{(y^2 - ax)^2}$$

$$y_2\left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)(-a-3a) - \left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)(-3a-a)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$$

$$= \frac{-4a - 4a}{3a^2/4} = \frac{-8a \times 4}{3a^2}$$

$$y_2 = \frac{-32}{3a}$$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{-32/3a}$$

$$= \frac{2\sqrt{2} \times 3a}{-32} = \frac{-3\sqrt{2}a}{16}$$

$$\boxed{|\rho| = \frac{3\sqrt{2}\,a}{16}}$$

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