



SNS COLLEGE OF TECHNOLOGY

Coimbatore-45
An Autonomous Institution



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Approved by AICTE, New Delhi & Affiliated to Anna University, Chennai

DEPARTMENT OF ELECTRONICS & COMMUNICATION ENGINEERING

19ECT212 – LINEAR CONTROL SYSTEMS

II YEAR/ IV SEMESTER

UNIT II – TIME RESPONSE ANALYSIS

TOPIC 4- IMPULSE AND STEP RESPONSE ANALYSIS OF SECOND ORDER SYSTEMS



OUTLINE



- REVIEW ABOUT PREVIOUS CLASS
- INTRODUCTION
- SECOND ORDER SYSTEM
- TIME-DOMAIN SPECIFICATION
- TRANSIENT RESPONSE ANALYSIS- RISE TIME, PEAK TIME, PERCENT OVERSHOOT, %OS, SETTING TIME
- ACTIVITY
- UNDERDAMPED-EXAMPLE
- OVERDAMPED RESPONSE
- STEP RESPONSE OF SECOND ORDER SYSTEM
- SUMMARY



SECOND – ORDER SYSTEM



- ***Second-order systems*** exhibit a wide range of responses which must be analyzed and described.
 - Whereas for a ***first-order system***, varying a single parameter changes the speed of response, **changes in the parameters of a *second order system*** can change the form of the response.
- ***For example:*** a second-order system can display characteristics much like a first-order system or, depending on component values, display damped or ***pure oscillations*** for its ***transient response***.



SECOND – ORDER SYSTEM



- A general second-order system is characterized by the following transfer function:

$$G(s) = \frac{b}{s^2 + as + b}$$

- We can re-write the above transfer function in the following form (closed loop transfer function):

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



SECOND – ORDER SYSTEM



ω_n ($\omega_n = \sqrt{b}$) - referred to as *the un-damped natural frequency* of the second order system, which is the frequency of oscillation of the system without damping.

ζ ($\zeta = \frac{a}{2\sqrt{b}}$) - referred to as *the damping ratio* of the second order system, which is a measure of the degree of resistance to change in the system output.

Poles;

$$\begin{aligned} & -\omega_n \zeta + \omega_n \sqrt{\zeta^2 - 1} \\ & -\omega_n \zeta - \omega_n \sqrt{\zeta^2 - 1} \end{aligned}$$

Poles are complex if $\zeta < 1$!



SECOND - ORDER SYSTEM



- According the value of ζ , a second-order system can be set into one of the four categories:

1. *Overdamped* - when the system has two real distinct poles ($\zeta > 1$).
2. *Underdamped* - when the system has two complex conjugate poles ($0 < \zeta < 1$)
3. *Undamped* - when the system has two imaginary poles ($\zeta = 0$).
4. *Critically damped* - when the system has two real but equal poles ($\zeta = 1$).



TIME-DOMAIN SPECIFICATION

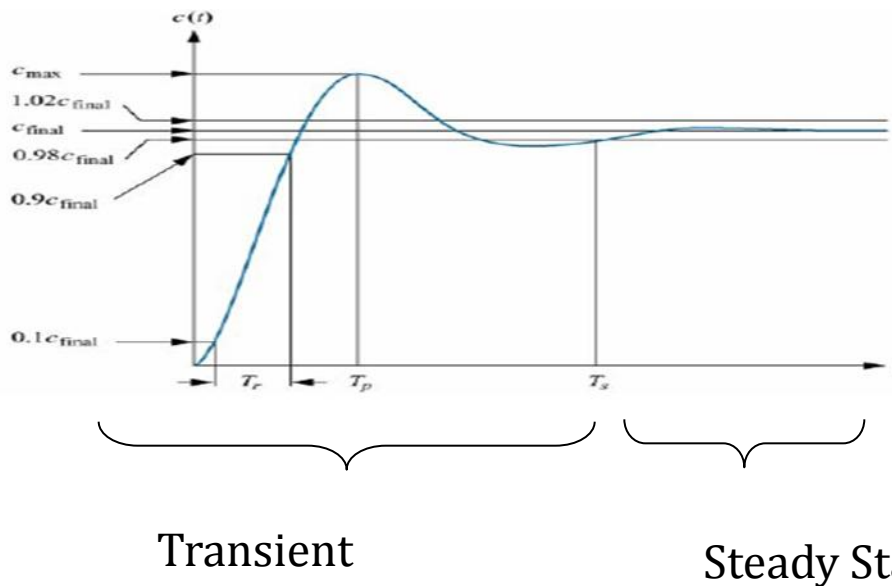


Given that the closed loop TF

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The system (2nd order system) is parameterized by ζ and ω_n

For $0 < \zeta < 1$ and $\omega_n > 0$, we like to investigate its response due to a unit step input



Two types of responses that are of interest:
(A) Transient response
(B) Steady state response



(A) For transient response 4 specifications:

$$(a) T_r - \text{rise time} = \frac{\pi - \theta}{\omega_n \sqrt{1 - \zeta^2}}$$

$$(b) T_p - \text{peak time} = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

$$(c) \%MP - \text{percentage maximum overshoot} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100\%$$

$$(d) T_s - \text{settling time (2\% error)} = \frac{4}{\zeta\omega_n}$$

(B) Steady State Response

(a) Steady State error-NEXT TOPIC...



Question : How are the performance related to ζ and ω_n ?



- Given a step input, i.e., $R(s) = 1/s$, then the system output (or step response) is;

$$C(s) = R(s)G(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

- Taking inverse Laplace transform, we have the step response;

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2} t + \theta\right)$$

Where;

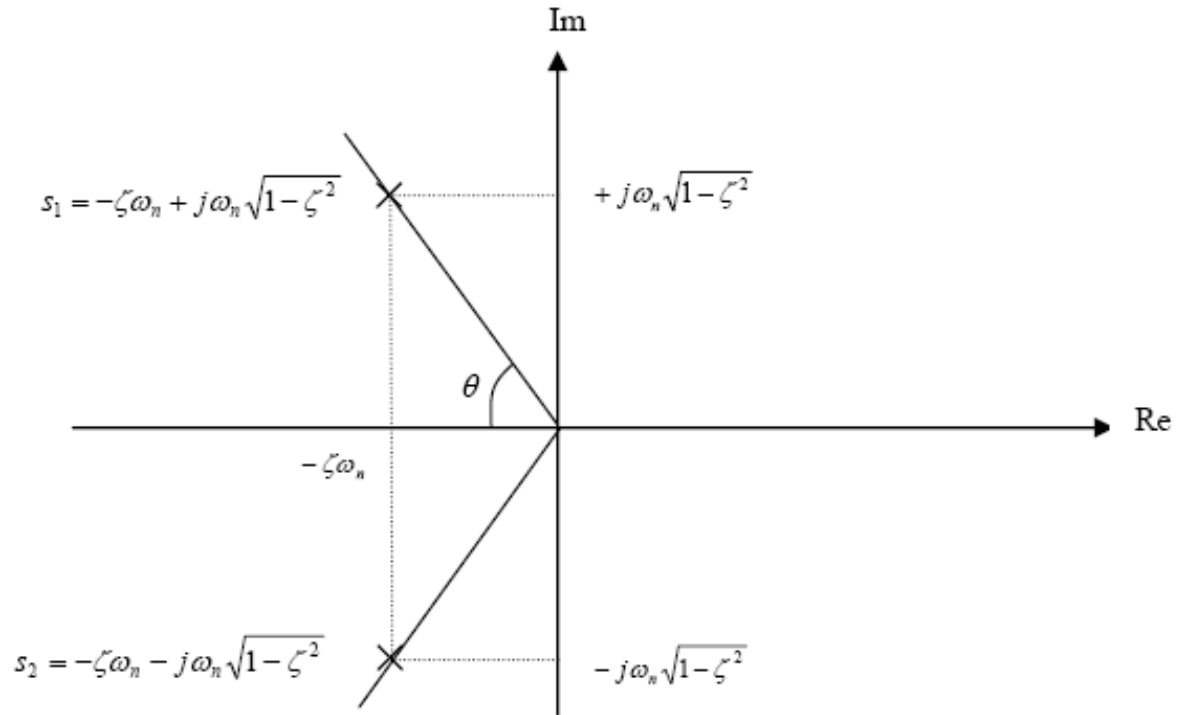
$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \quad \text{or} \quad \theta = \cos^{-1}(\zeta)$$



SECOND - ORDER SYSTEM



$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



$$\theta = \tan^{-1} \left(\frac{\sqrt{1-\zeta^2}}{\zeta} \right)$$

Mapping the poles into s-plane



Lets re-write the equation for $c(t)$:

Let: $\beta = \sqrt{1 - \xi^2}$

and

$$\omega_d = \omega_n \sqrt{1 - \xi^2} \quad \left. \vphantom{\omega_d} \right\} \begin{array}{l} \text{Damped natural frequency} \\ \omega_n > \omega_d \end{array}$$

Thus:

$$c(t) = 1 - \frac{1}{\beta} e^{-\xi \omega_n t} \sin(\omega_d t + \theta)$$

where $\theta = \cos^{-1}(\xi)$



TRANSIENT RESPONSE ANALYSIS



1) *Rise time, T_r* . Time the response takes to rise from 0 to 100%

$$c(t) \Big|_{t=T_r} = 1 - \underbrace{\frac{1}{\beta}}_{\neq 0} e^{-\xi\omega_n t} \underbrace{\sin(\omega_d t + \theta)}_{=0} = 1$$

$$\sin(\omega_d T_r + \theta) = 0$$

$$\omega_d T_r + \theta = \sin^{-1}(0) = \pi$$

$$T_r = \frac{\pi - \theta}{\omega_n \sqrt{1 - \xi^2}}$$



TRANSIENT RESPONSE ANALYSIS



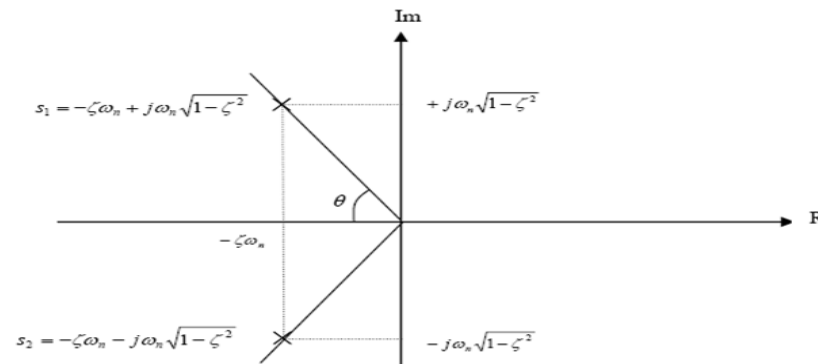
2) *Peak time, T_p* - The peak time is the time required for the response to reach the first peak, which is given by;

$$\left. \dot{c}(t) \right|_{t=T_p} = 0$$

$$\left. \dot{c}(t) \right|_{t=T_p} = -\frac{1}{\beta} (-\zeta\omega_n) e^{-\zeta\omega_n t} \sin(\omega_d t + \theta) - \frac{1}{\beta} e^{-\zeta\omega_n t} \cos(\omega_d t + \theta) \left[\omega_n \sqrt{1-\zeta^2} \right] = 0$$

$$\frac{\zeta\omega_n}{\beta} e^{-\zeta\omega_n T_p} \sin(\omega_d T_p + \theta) = \frac{\left[\omega_n \sqrt{1-\zeta^2} \right]}{\beta} e^{-\zeta\omega_n T_p} \cos(\omega_d T_p + \theta)$$

$$\tan(\omega_d T_p + \theta) = \frac{\sqrt{1-\zeta^2}}{\zeta}$$



$$\tan \theta = \frac{\sqrt{1-\zeta}}{\zeta}$$



TRANSIENT RESPONSE ANALYSIS



3) *Percent overshoot, %OS* - The percent overshoot is defined as the amount that the waveform at the peak time overshoots the steady-state value, which is expressed as a percentage of the steady-state value.

$$\% MP \equiv \frac{C(T_p) - C(\infty)}{C(\infty)} \times 100\%$$

OR
$$\% OS = \frac{C_{\max} - C_{\text{final}}}{C_{\text{final}}} \times 100$$



We know that $\tan(\theta) = \tan(\pi + \theta)$

So, $\tan(\omega_d T_p + \theta) = \tan(\pi + \theta)$

From this expression:

$$\omega_d T_p + \theta = \pi + \theta$$

$$\omega_d T_p = \pi$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$



$$\begin{aligned}\frac{C(T_p) - 1}{1} x100\% &= -\frac{1}{\beta} e^{-\xi\omega_n t} \sin(\omega_d t + \theta) x100\% \\ &= -\frac{1}{\beta} e^{-\xi\omega_n \left[\frac{\pi}{\omega_n \sqrt{1-\zeta^2}} \right]} \sin\left(\omega_d \left(\frac{\pi}{\omega_d} \right) + \theta\right) x100\% \\ &= -\frac{1}{\beta} e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \sin(\pi + \theta) x100\% \\ &= \frac{\sin(\theta)}{\beta} e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} x100\% = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} x100\%\end{aligned}$$

$$\beta = \sqrt{1 - \xi^2}$$

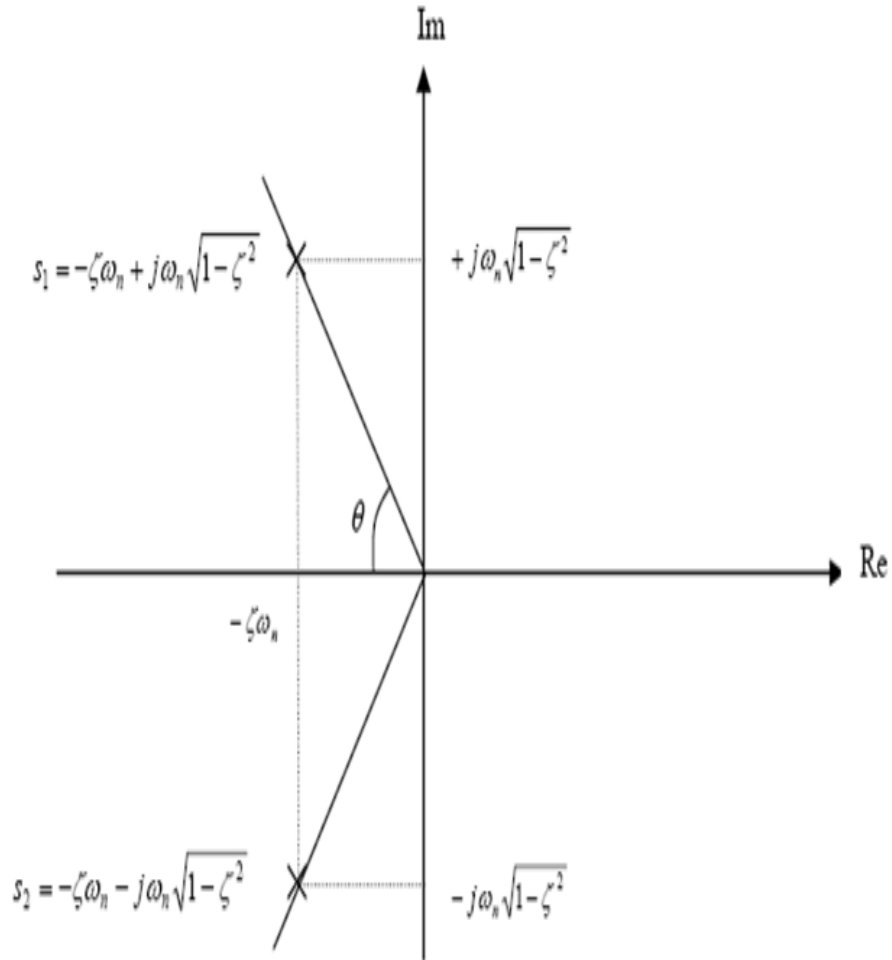
$$\sin \theta = \sqrt{1 - \zeta^2}$$



ACTIVITY-GD

Therefore,

$$\%MP = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100\%$$



- For given %OS, the damping ratio can be solved from the above equation;

$$\zeta = \frac{-\ln(\%MP / 100)}{\sqrt{\pi^2 + \ln^2(\%MP / 100)}}$$



TRANSIENT RESPONSE ANALYSIS



4) *Setting time, T_s* - The settling time is the time required for the amplitude of the sinusoid to decay to 2% of the steady-state value.

To find T_s , we must find the time for which $c(t)$ reaches & stays within $\pm 2\%$ of the steady state value, c_{final} . The settling time is the time it takes for the amplitude of the decaying sinusoid in $c(t)$ to reach 0.02, or

$$e^{-\zeta\omega_n T_s} \frac{1}{\sqrt{1-\zeta^2}} = 0.02$$

Thus,

$$T_s = \frac{4}{\zeta\omega_n}$$



UNDERDAMPED



Example 2: Find the natural frequency and damping ratio for the system with transfer function

$$G(s) = \frac{36}{s^2 + 4.2s + 36}$$

Solution:

Compare with general TF.

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

• $\omega_n = 6$

• $\zeta = 0.35$



UNDERDAMPED

Example 3: Given the transfer function

$$G(s) = \frac{100}{s^2 + 15s + 100}$$

find T_s , %OS, T_p

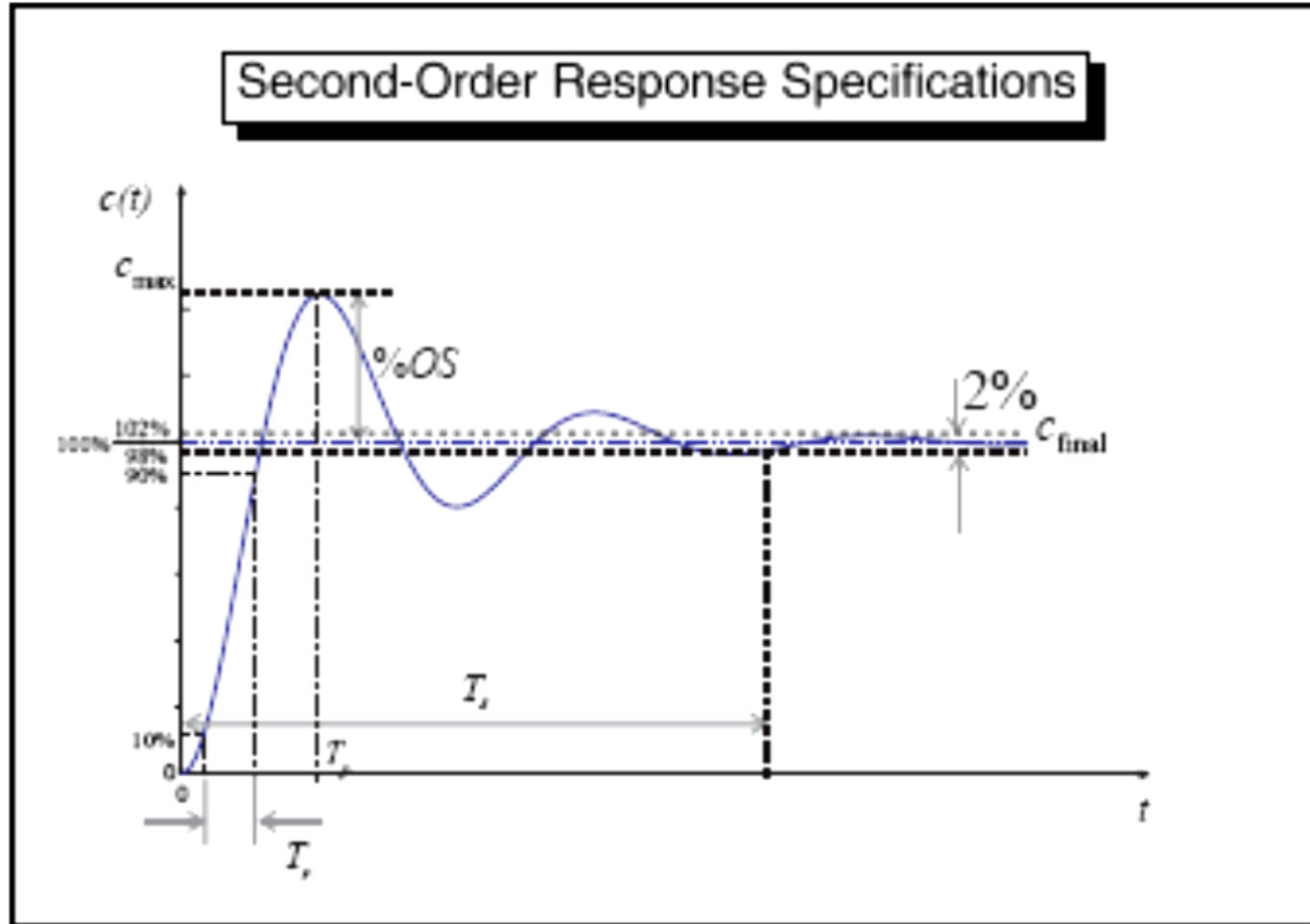
Solution:

$$\omega_n = 10 \quad \xi = 0.75$$

$$T_s = 0.533s, \text{ \%OS} = 2.838\%, T_p = 0.475s$$



UNDERDAMPED



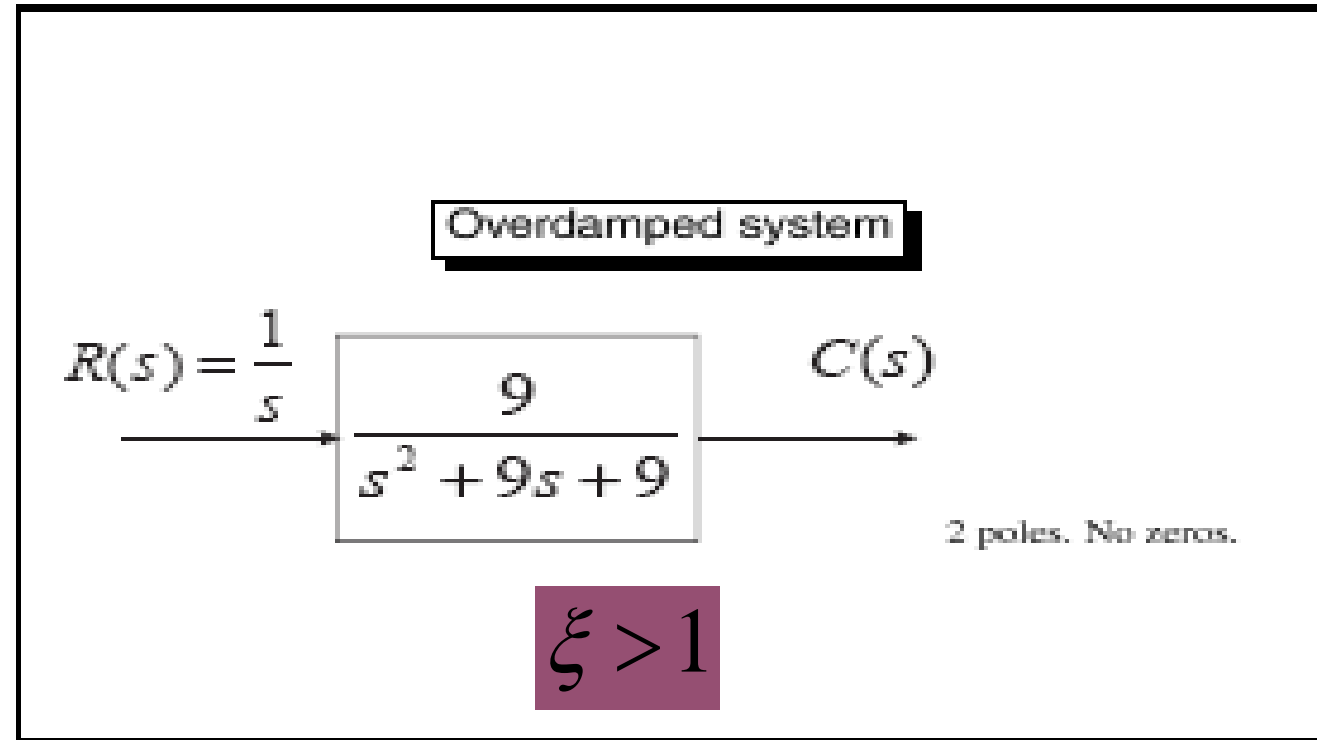


OVERDAMPED RESPONSE

$$G(s) = \frac{b}{s^2 + as + b}$$



$$a = 9$$



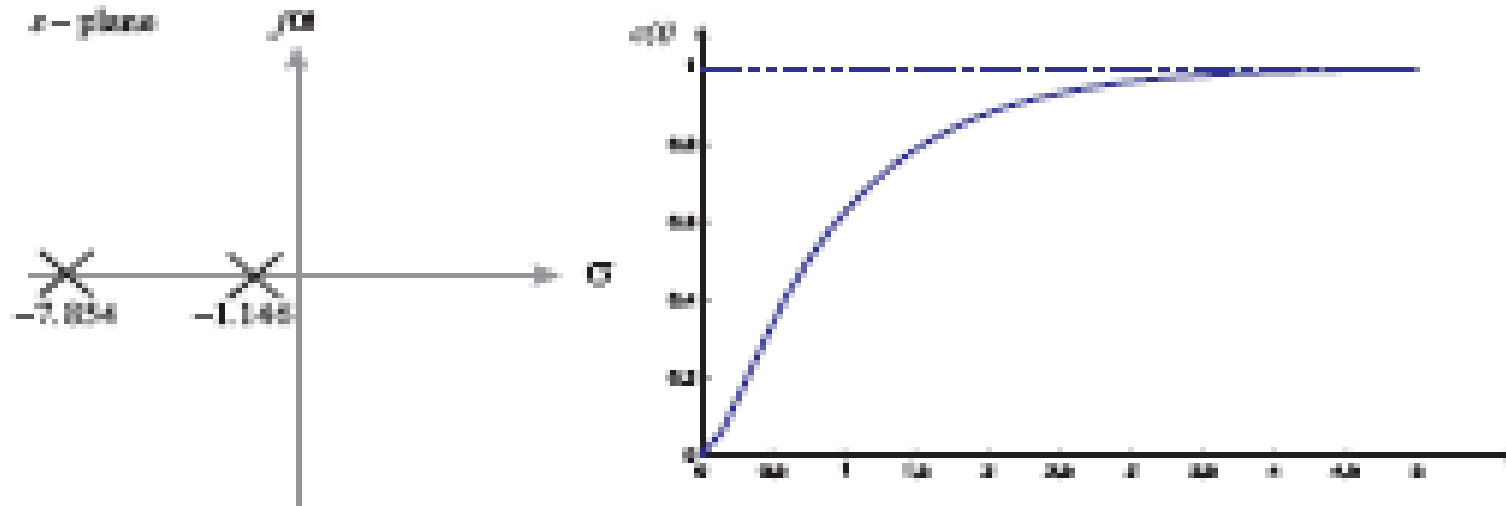
$$C(s) = \frac{9}{s(s^2 + 9s + 9)} = \frac{9}{s(s + 7.854)(s + 1.146)}$$

$s = 0; s = -7.854; s = -1.146$ (two real poles)



$$c(t) = K_1 + K_2 e^{-7.854t} + K_3 e^{-1.146t}$$

Overdamped response



OVERDAMPED RESPONSE !!!

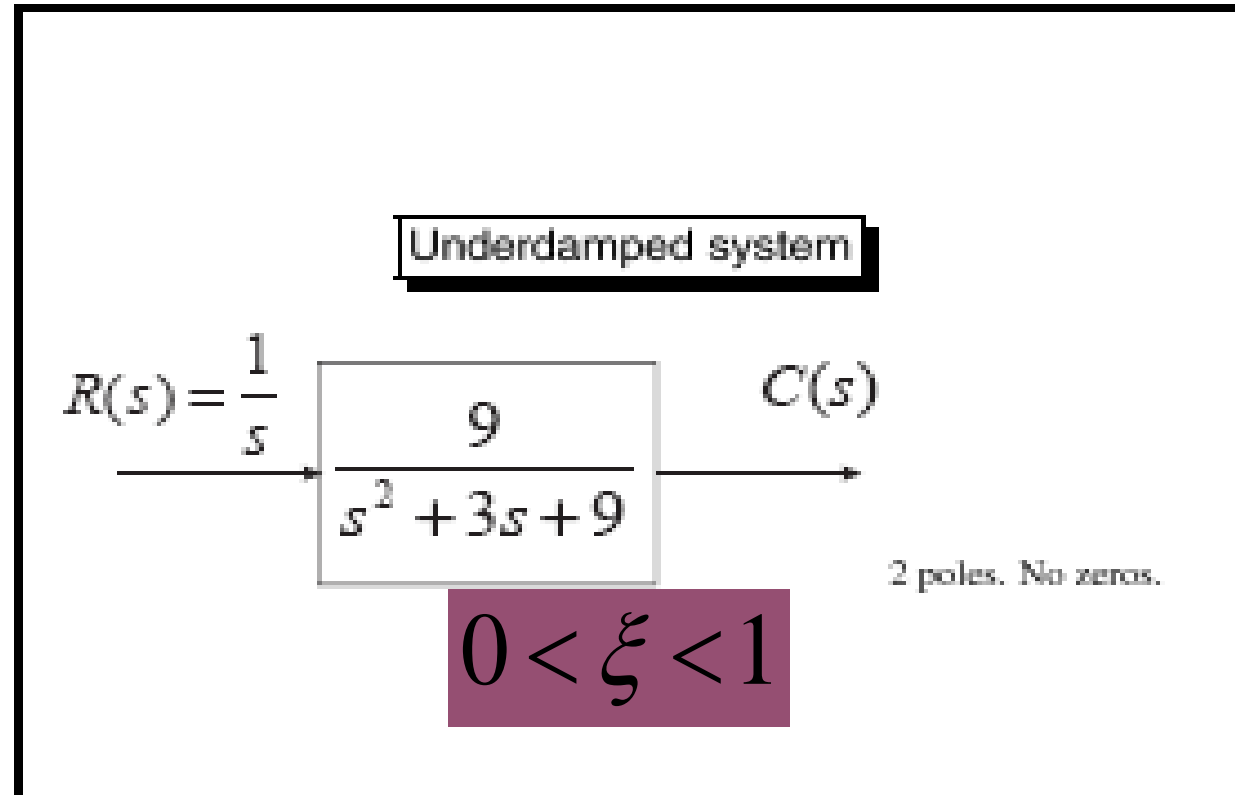


Underdamped Response

$$G(s) = \frac{b}{s^2 + as + b}$$



$$a = 3$$

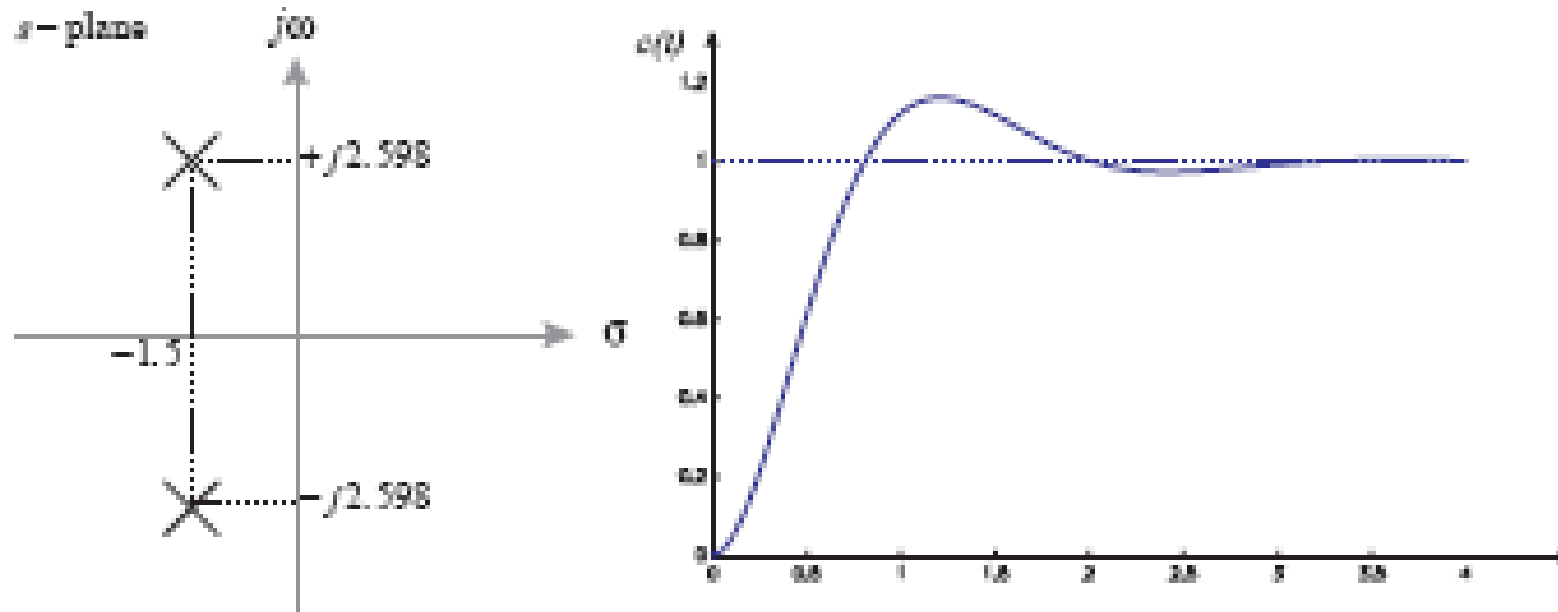


$$c(t) = K_1 + e^{-1.5t} (K_2 \cos 2.598t + K_3 \sin 2.598t)$$

$$s = 0; s = -1.5 \pm j2.598 \text{ (two complex poles)}$$



Underdamped response



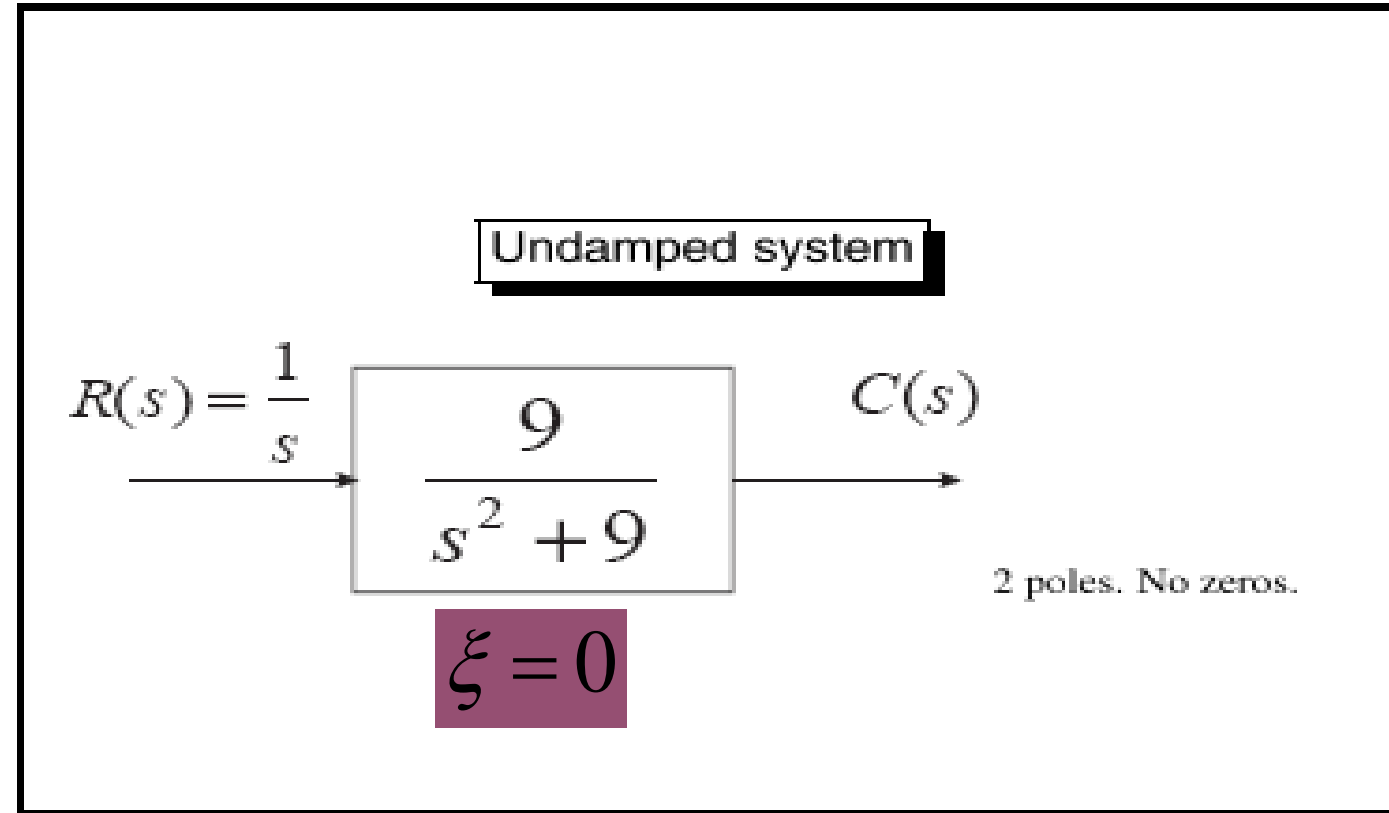
UNDERDAMPED RESPONSE !!!



Undamped Response

$$G(s) = \frac{b}{s^2 + as + b}$$

$$a = 0$$

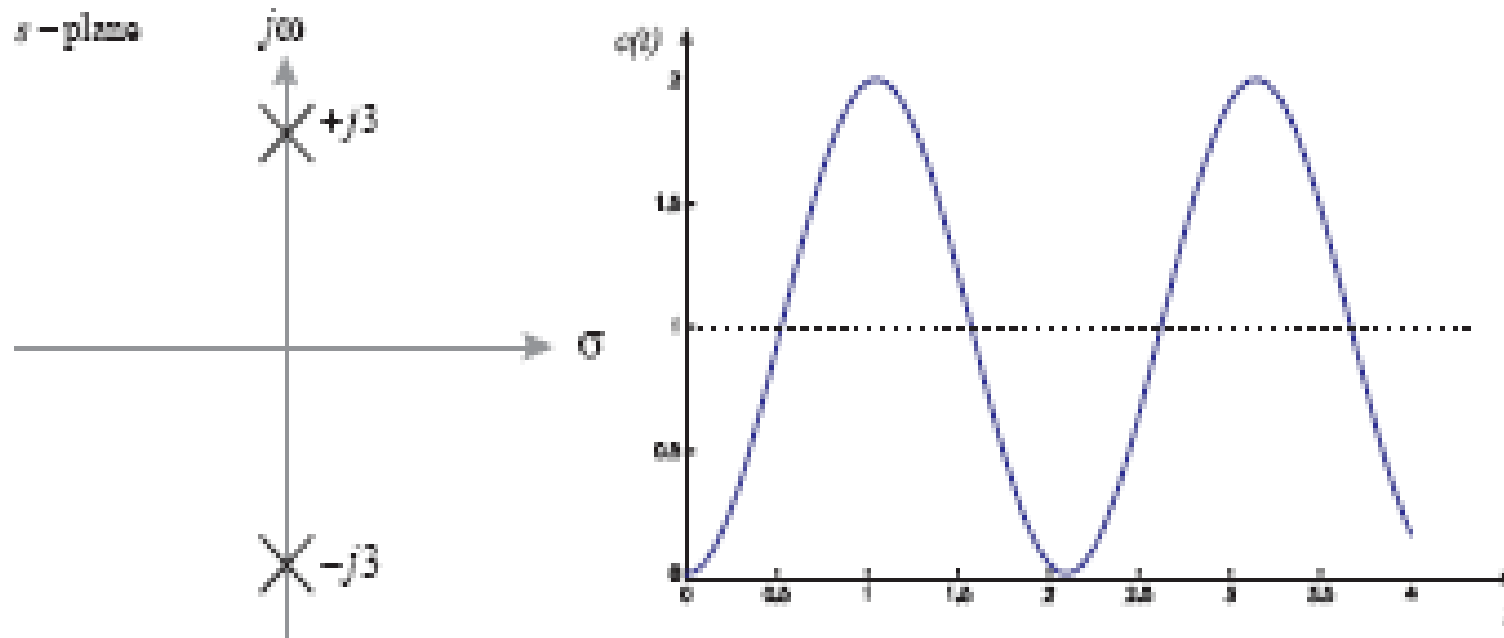


$$c(t) = K_1 + K_2 \cos 3t$$

$$s = 0; s = \pm j3 \text{ (two imaginary poles)}$$



Undamped response



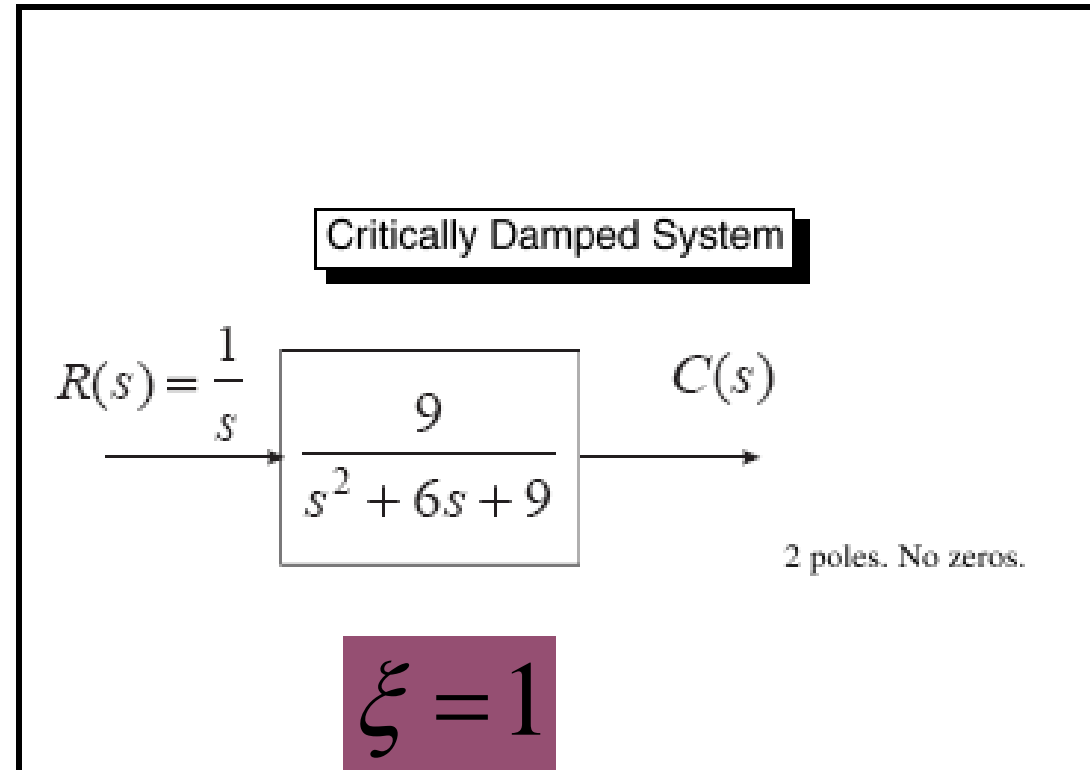


Critically Damped System

$$G(s) = \frac{b}{s^2 + as + b}$$



$$a = 6$$

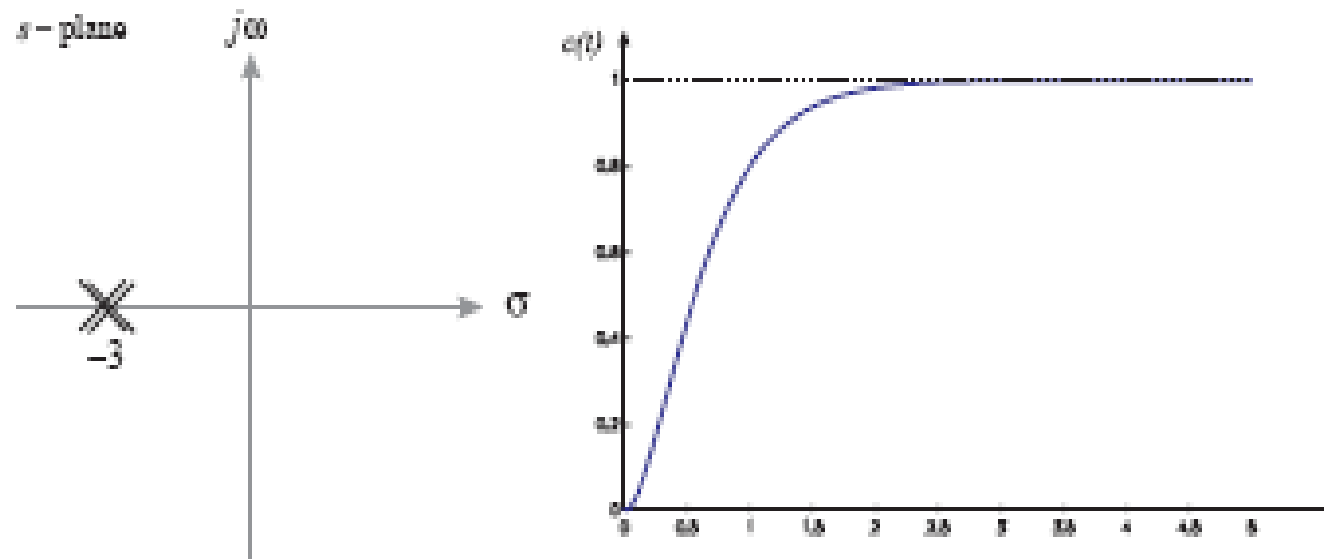


$$c(t) = K_1 + K_2 e^{-3t} + K_3 t e^{-3t}$$

$S = 0; s = -3, -3$ (two real and equal poles)

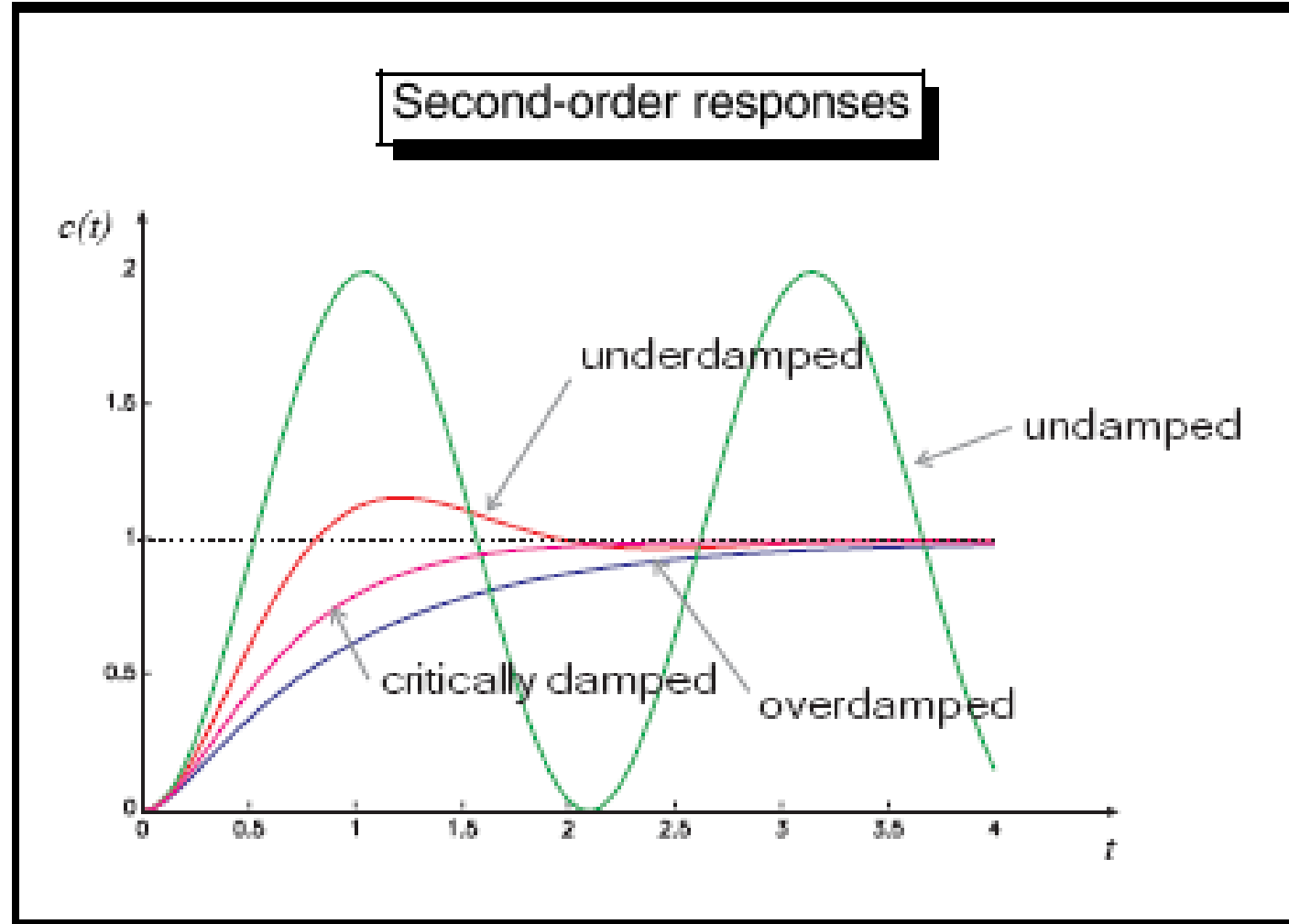


Critically Damped Response





SECOND - ORDER SYSTEM

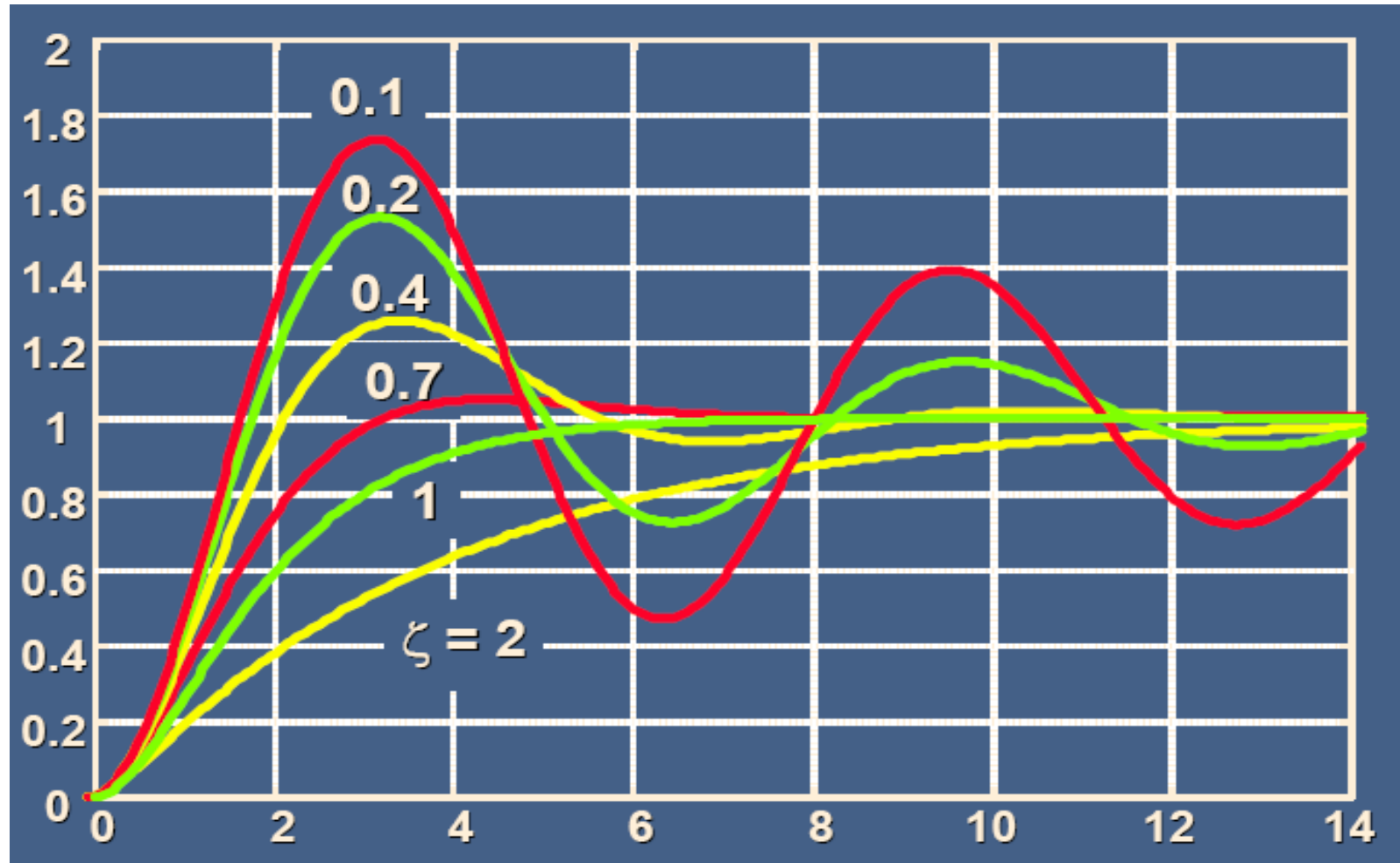




System	Pole-zero Plot	Response
<p>(a) $R(s) = \frac{1}{s}$ \rightarrow $G(s) = \frac{b}{s^2 + as + b}$ \rightarrow $C(s)$</p> <p>General</p>		
<p>(b) $R(s) = \frac{1}{s}$ \rightarrow $G(s) = \frac{9}{s^2 + 9s + 9}$ \rightarrow $C(s)$</p> <p>Overdamped</p>		
<p>(c) $R(s) = \frac{1}{s}$ \rightarrow $G(s) = \frac{9}{s^2 + 2s + 9}$ \rightarrow $C(s)$</p> <p>Underdamped</p>		
<p>(d) $R(s) = \frac{1}{s}$ \rightarrow $G(s) = \frac{9}{s^2 + 9}$ \rightarrow $C(s)$</p> <p>Undamped</p>		
<p>(e) $R(s) = \frac{1}{s}$ \rightarrow $G(s) = \frac{9}{s^2 + 6s + 9}$ \rightarrow $C(s)$</p> <p>Critically damped</p>		



EFFECT OF DIFFERENT DAMPING RATIO, ξ



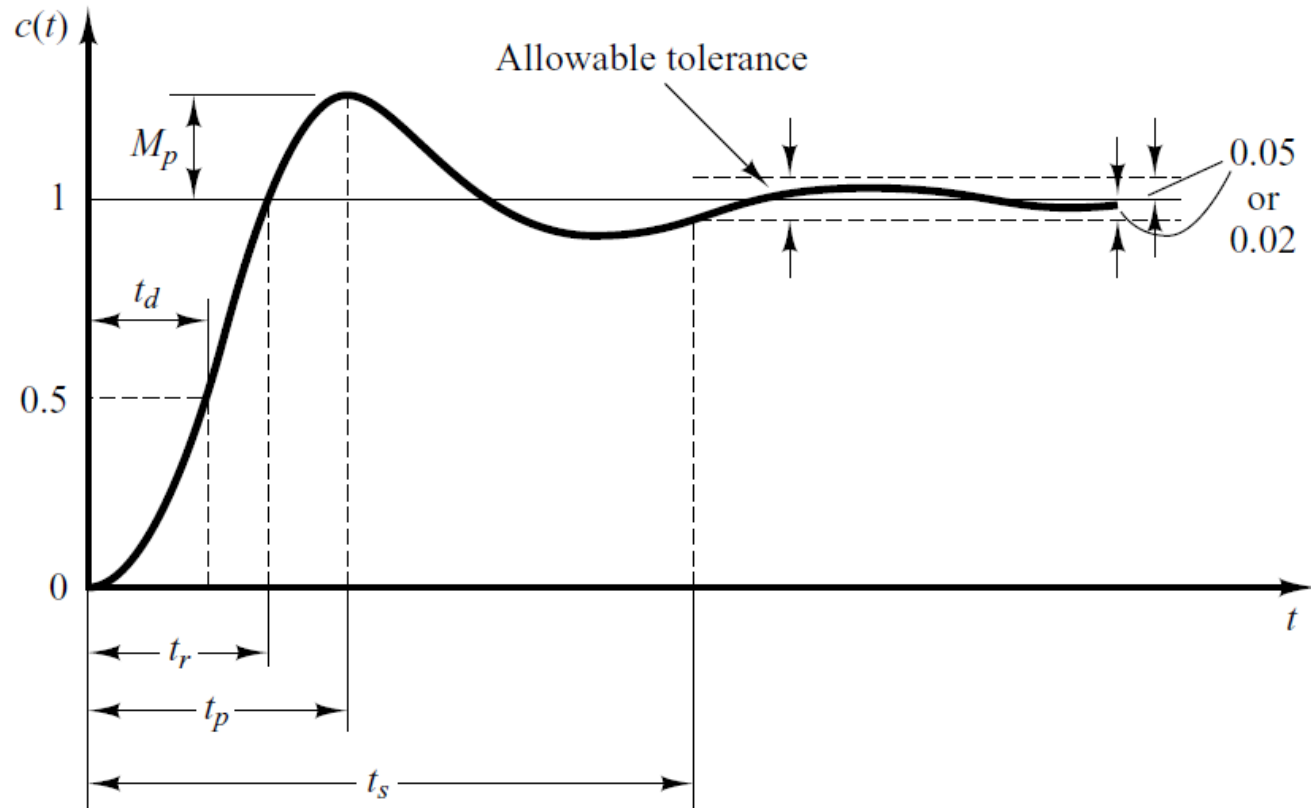


UNDERDAMPED SYSTEM



For $0 < \zeta < 1$ and $\omega_n > 0$, the 2nd order system's response due to a unit step input:

Important timing characteristics: delay time, rise time, peak time, maximum overshoot, and settling time.

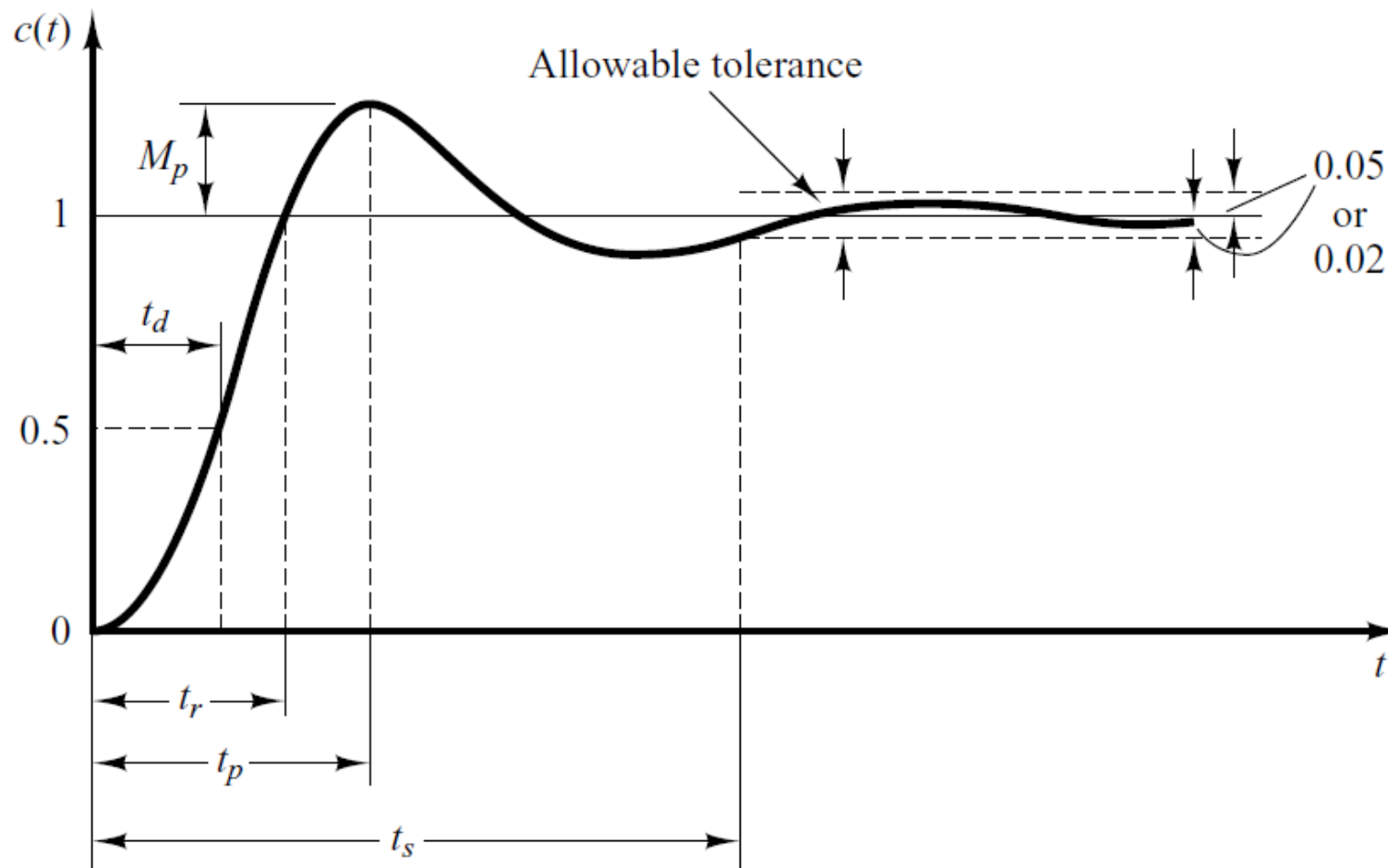




DELAY TIME



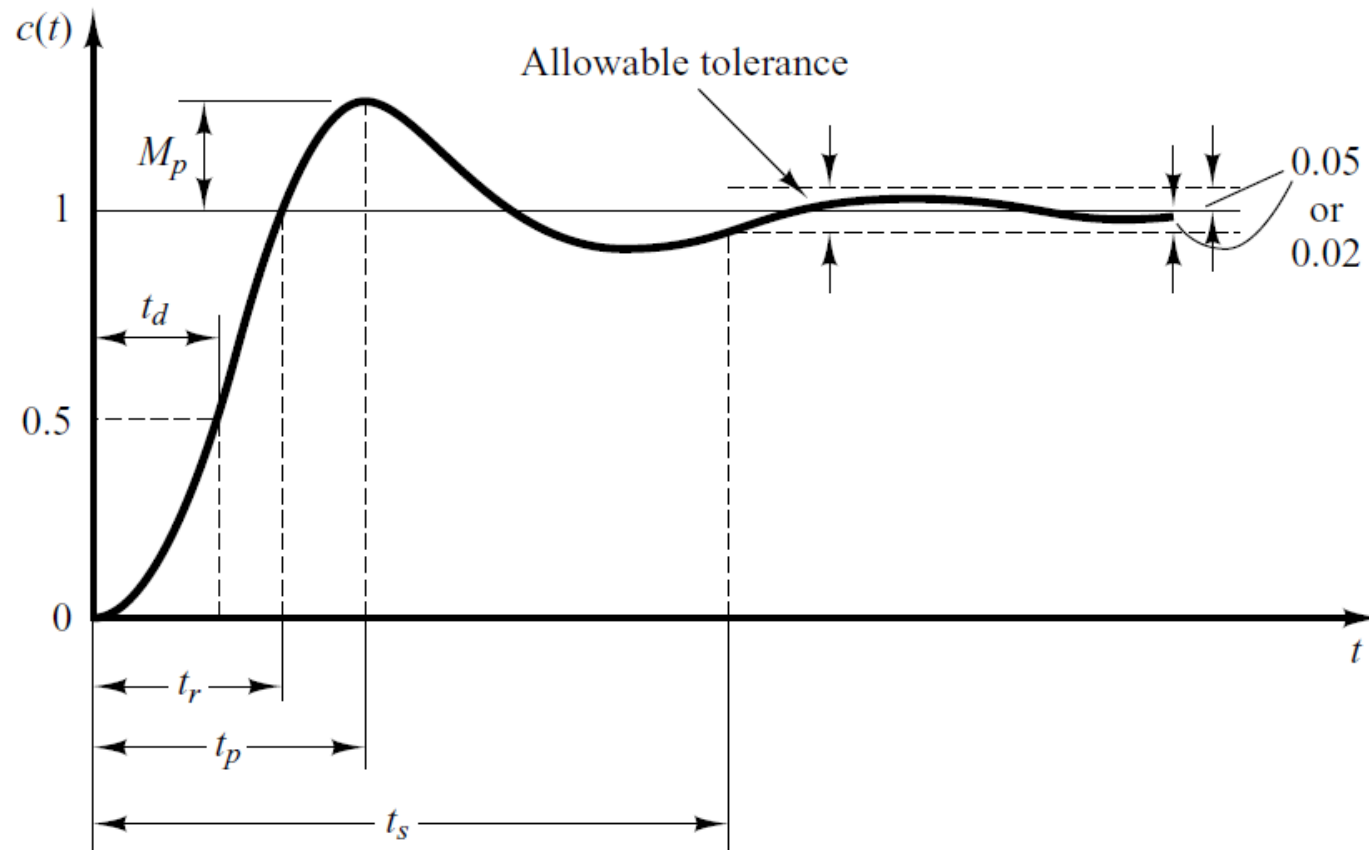
- The delay (t_d) time is the time required for the response to reach half the final value the very first time.





Rise Time

- The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value.
- For underdamped second order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.

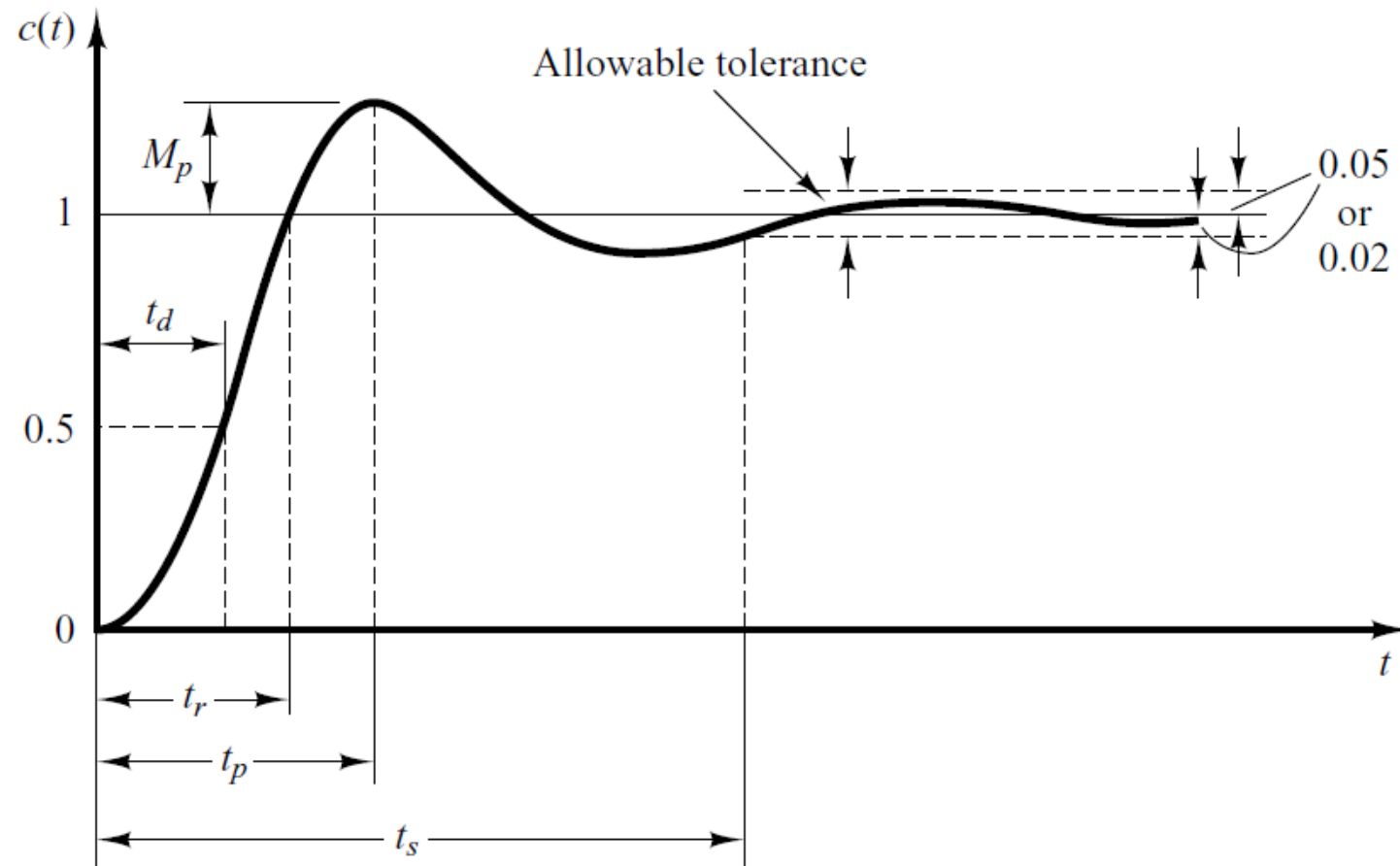




PEAK TIME



- The peak time is the time required for the response to reach the first peak of the overshoot.





Maximum Overshoot



The maximum overshoot is the maximum peak value of the response curve measured from unity.

If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

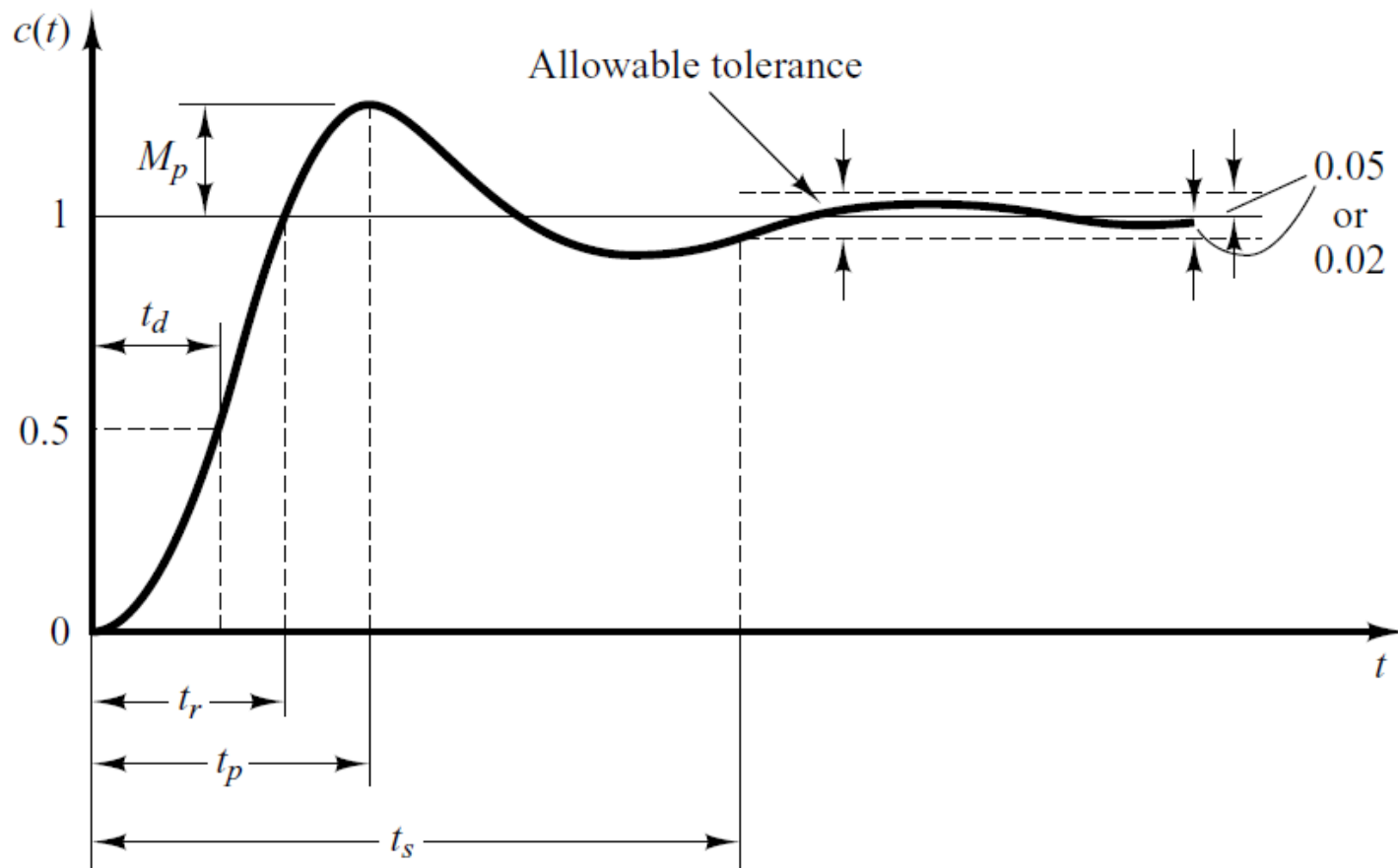
The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.



Settling Time



- The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).





STEP RESPONSE OF UNDERDAMPED SYSTEM



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \xrightarrow{\text{Step Response}} C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

- The partial fraction expansion of above equation is given as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2}$$

Annotations: A blue arrow points from the term $\omega_n^2(1 - \zeta^2)$ to the constant term in the denominator. Another blue arrow points from the term $(s + 2\zeta\omega_n)^2$ to the denominator.

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$



STEP RESPONSE OF UNDERDAMPED SYSTEM



$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

- Above equation can be written as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

- Where $\omega_d = \omega_n \sqrt{1 - \zeta^2}$, is the frequency of transient oscillations and is called **damped natural frequency**.
- The inverse Laplace transform of above equation can be obtained easily if **C(s)** is written in the following form:

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$



STEP RESPONSE OF UNDERDAMPED SYSTEM



$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\frac{\zeta}{\sqrt{1-\zeta^2}} \omega_n \sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta}{\sqrt{1-\zeta^2}} \frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$c(t) = 1 - e^{-\zeta\omega_n t} \cos \omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_d t$$



STEP RESPONSE OF UNDERDAMPED SYSTEM



$$c(t) = 1 - e^{-\zeta\omega_n t} \cos \omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_d t$$

$$c(t) = 1 - e^{-\zeta\omega_n t} \left[\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right]$$

- When $\zeta = 0$

$$\begin{aligned}\omega_d &= \omega_n \sqrt{1-\zeta^2} \\ &= \omega_n\end{aligned}$$

$$c(t) = 1 - \cos \omega_n t$$

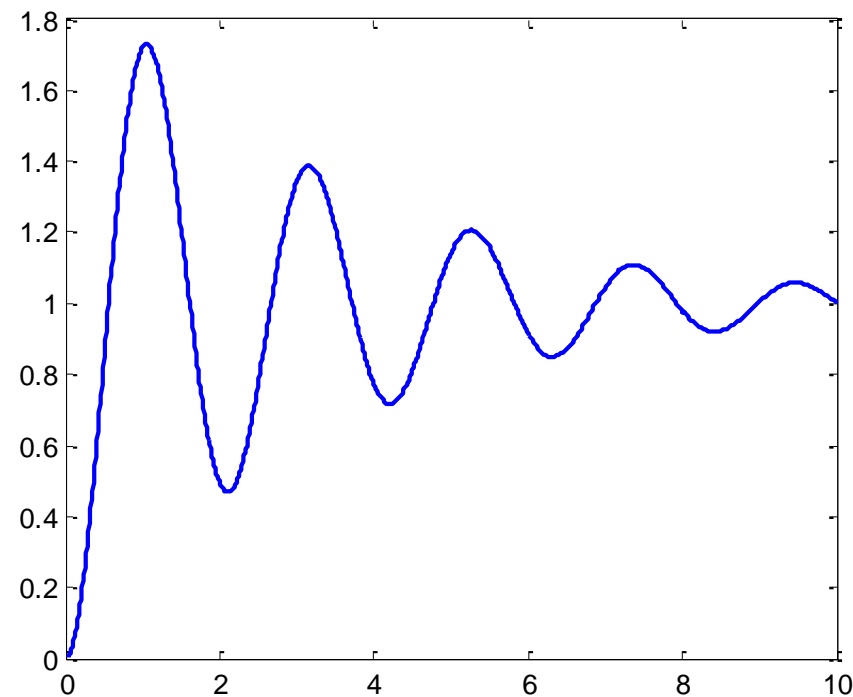


STEP RESPONSE OF UNDERDAMPED SYSTEM



$$c(t) = 1 - e^{-\zeta\omega_n t} \left[\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right]$$

if $\zeta = 0.1$ and $\omega_n = 3$



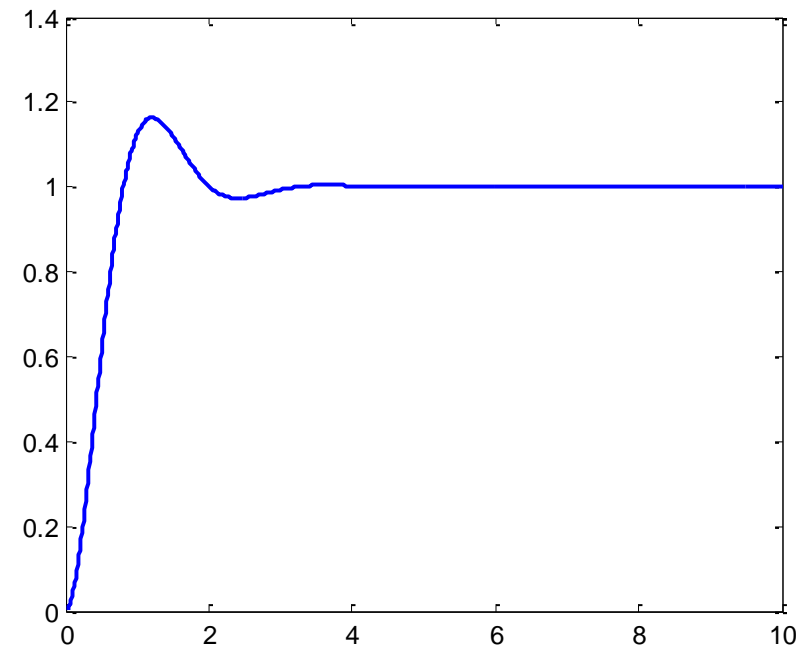


STEP RESPONSE OF UNDERDAMPED SYSTEM



$$c(t) = 1 - e^{-\zeta\omega_n t} \left[\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right]$$

if $\zeta = 0.5$ and $\omega_n = 3$





SUMMARY

