



SNS COLLEGE OF TECHNOLOGY

Coimbatore-28
An Autonomous Institution



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Approved by AICTE, New Delhi & Affiliated to Anna University, Chennai

DEPARTMENT OF ELECTRONICS & COMMUNICATION ENGINEERING

19ECT212 – LINEAR CONTROL SYSTEMS

II YEAR/ IV SEMESTER

UNIT II – TIME RESPONSE ANALYSIS

TOPIC 3- IMPULSE AND STEP RESPONSE ANALYSIS OF SECOND ORDER SYSTEMS



OUTLINE



- REVIEW ABOUT PREVIOUS CLASS
- INTRODUCTION
- SECOND ORDER SYSTEM
- STEP RESPONSE OF SECOND ORDER SYSTEM
- ACTIVITY
- IMPULSE RESPONSE OF SECOND ORDER SYSTEM
- DAMPING RATIO
 - 1. Undamped system, $\zeta = 0$
 - 2. Under damped system, $0 < \zeta < 1$
 - 3. Critically damped system, $\zeta = 1$
 - 4. Over damped system, $\zeta > 1$
- SUMMARY



Second Order System

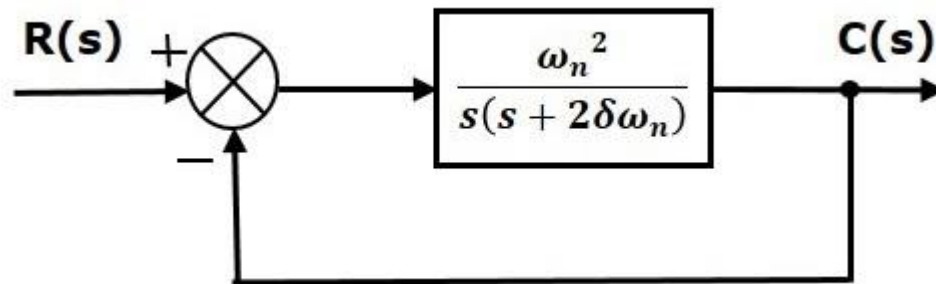


- the affect of location of poles and zeros on the transient response of 1st order systems already discussed
- Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described.
- Varying a first-order system's parameter (T, K) simply changes the speed and offset of the response
- Whereas, changes in the parameters of a second-order system can change the *form of* the response.
- A second-order system can display characteristics much like a first-order system or, depending on component values, display damped or pure oscillations for its *transient response*.



INTRODUCTION

- Consider the following block diagram of closed loop control system. Here, an open loop transfer function, $\frac{\omega_n^2}{s(s+2\zeta\omega_n)}$ is connected with a unity negative feedback.



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$



INTRODUCTION



- ω_n \longrightarrow **un-damped natural frequency** of the second order system, which is the frequency of oscillation of the system without damping.
- ζ \longrightarrow **damping ratio** of the second order system, which is a measure of the degree of resistance to change in the system output.



- Determine the un-damped natural frequency and damping ratio of the following second order system.

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4}$$

- Compare the numerator and denominator of the given transfer function with the general 2nd order transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 4 \quad \Rightarrow \omega_n = 2$$

$$\Rightarrow 2\zeta\omega_n s = 2s$$

$$\Rightarrow \zeta\omega_n = 1$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2s + 4$$

$$\Rightarrow \zeta = 0.5$$



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Two poles of the system are

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

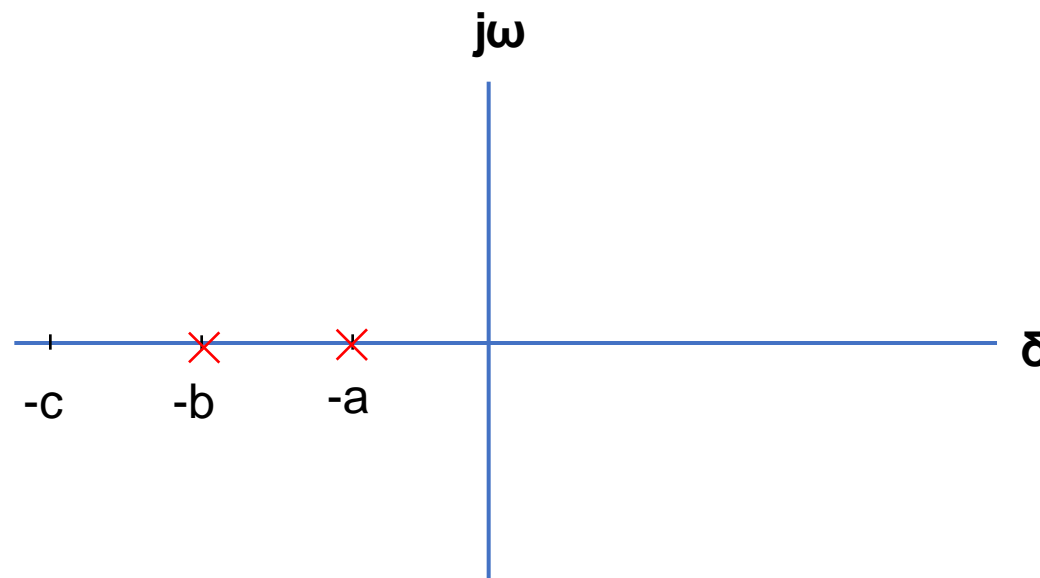
$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$



$$-\omega_n \zeta + \omega_n \sqrt{\zeta^2 - 1}$$
$$-\omega_n \zeta - \omega_n \sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories :

1. *Overdamped* - when the system has two real distinct poles ($\zeta > 1$).

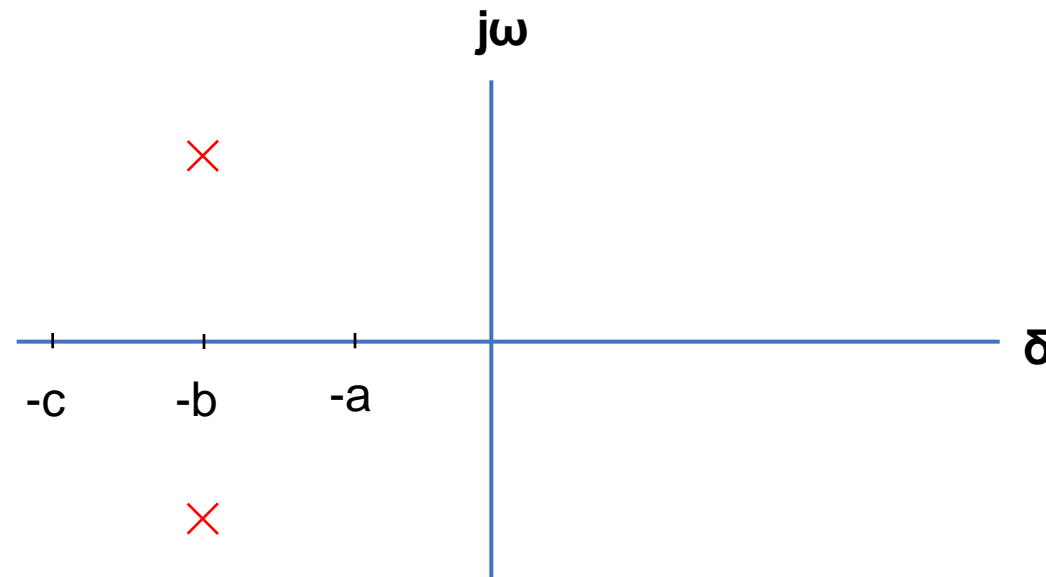




$$-\omega_n \zeta + \omega_n \sqrt{\zeta^2 - 1}$$
$$-\omega_n \zeta - \omega_n \sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories

2. *Underdamped* - when the system has two complex conjugate poles ($0 < \zeta < 1$)



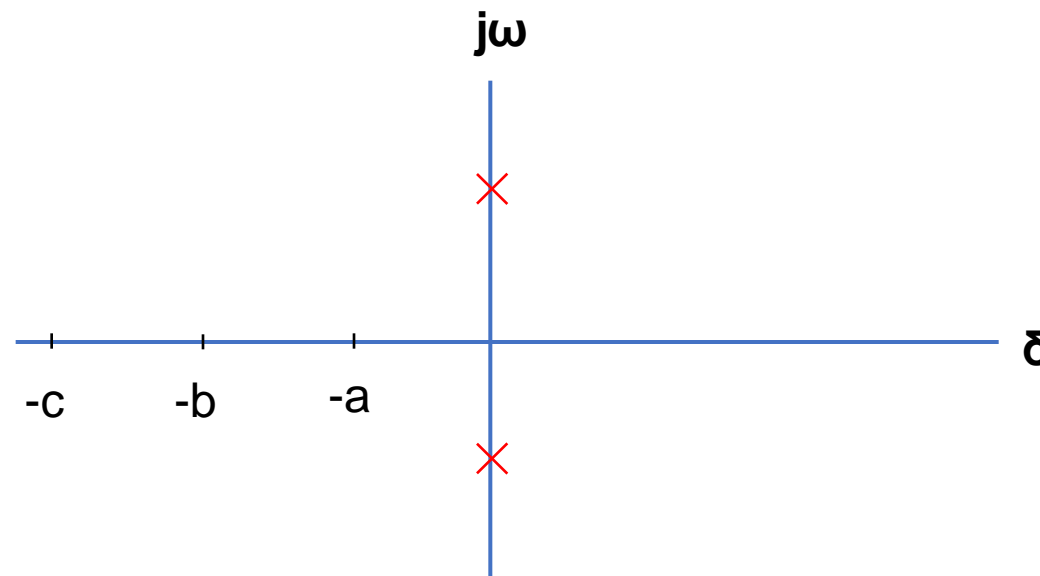


$$-\omega_n \zeta + \omega_n \sqrt{\zeta^2 - 1}$$

$$-\omega_n \zeta - \omega_n \sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories

3. *Undamped* - when the system has two imaginary poles ($\zeta = 0$).

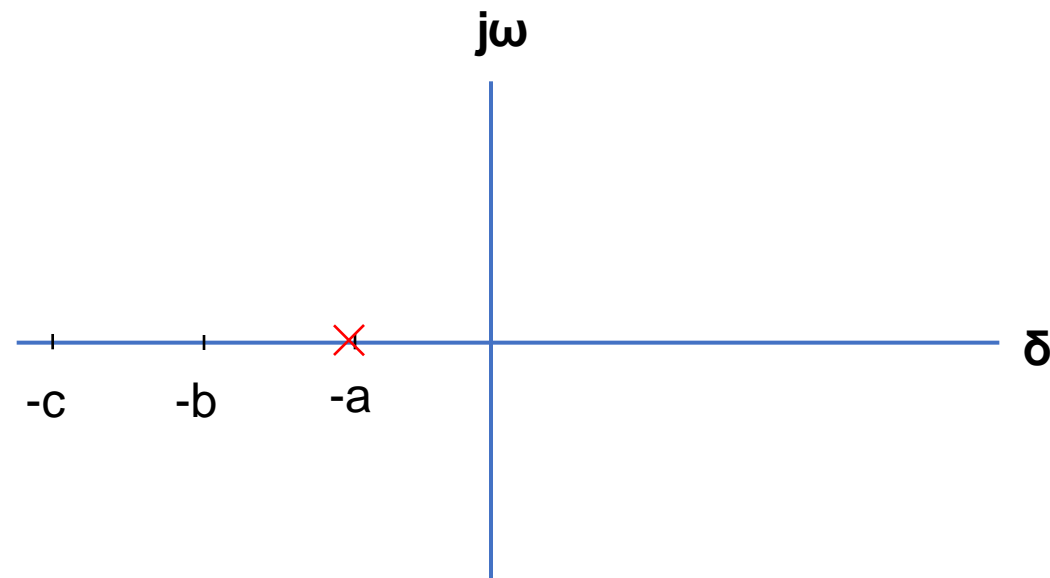




$$-\omega_n \zeta + \omega_n \sqrt{\zeta^2 - 1}$$
$$-\omega_n \zeta - \omega_n \sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories

4. *Critically damped* - when the system has two real but equal poles ($\zeta = 1$).





SECOND ORDER SYSTEM

Transfer function of the closed loop control system having unity negative feedback as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

Substitute, $G(s) = \frac{\omega_n^2}{s(s+2\delta\omega_n)}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)}{1 + \left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

The power of 's' is two in the denominator term. Hence, the above transfer function is of the second order and the system is said to be the **second order system**.



The characteristic equation is -

$$s^2 + 2\delta\omega_n s + \omega_n^2 = 0$$

The roots of characteristic equation are -

$$s = \frac{-2\delta\omega_n \pm \sqrt{(2\delta\omega_n)^2 - 4\omega_n^2}}{2} = \frac{-2(\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1})}{2}$$

$$\Rightarrow s = -\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1}$$

- The two roots are imaginary when $\delta = 0$.
- The two roots are real and equal when $\delta = 1$.
- The two roots are real but not equal when $\delta > 1$.
- The two roots are complex conjugate when $0 < \delta < 1$.

We can write $C(s)$ equation as,

$$C(s) = \left(\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2} \right) R(s)$$



Where,

$C(s)$ is the Laplace transform of the output signal, $c(t)$

$R(s)$ is the Laplace transform of the input signal, $r(t)$

ω_n is the natural frequency

δ is the damping ratio.

Follow these steps to get the response (output) of the second order system in the time domain

- Take Laplace transform of the input signal, $r(t)$.
- Consider the equation,

$$C(s) = \left(\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2} \right) R(s)$$

- Substitute $R(s)$ value in the above equation.
- Do partial fractions of $C(s)$ if required.
- Apply inverse Laplace transform to $C(s)$.



STEP RESPONSE OF SECOND ORDER SYSTEM

Consider the unit step signal as an input to the second order system.

Laplace transform of the unit step signal is,

$$R(s) = 1 / s$$

We know the transfer function of the second order closed loop control system is,



STEP RESPONSE ...

the transfer function of the second order closed loop control system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Case 1: $\delta = 0$

Substitute, $\delta = 0$ in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2} \right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2} \right) \left(\frac{1}{s} \right) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)}$$



STEP RESPONSE...



Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - \cos(\omega_n t))u(t)$$

unit step response of the second order system when $\zeta = 0$ will be a continuous time signal with constant amplitude and frequency.



STEP RESPONSE...



Case 2: $\delta = 1$

Substitute, $\delta = 1$ in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s + \omega_n)^2} \right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s + \omega_n)^2} \right) \left(\frac{1}{s} \right) = \frac{\omega_n^2}{s(s + \omega_n)^2}$$

$$C(s) = \frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t})u(t)$$



ACTIVITY



GROUP DISCUSSION



IMPULSE RESPONSE OF SECOND ORDER SYSTEM



The **impulse response** of the second order system can be obtained by using any one of these two methods.

- Follow the procedure involved while deriving step response by considering the value of $R(s)$ as 1 instead of $1/s$.
- Do the differentiation of the step response.

The following table shows the impulse response of the second order system for 4 cases of the damping ratio.



Condition of Damping ratio	Impulse response for $t \geq 0$
$\delta = 0$	$\omega_n \sin(\omega_n t)$
$\delta = 1$	$\omega_n^2 t e^{-\omega_n t}$
$0 < \delta < 1$	$\left(\frac{\omega_n e^{-\delta \omega_n t}}{\sqrt{1-\delta^2}} \right) \sin(\omega_d t)$
$\delta > 1$	$\left(\frac{\omega_n}{2\sqrt{\delta^2-1}} \right) \left(e^{-(\delta\omega_n - \omega_n \sqrt{\delta^2-1})t} - e^{-(\delta\omega_n + \omega_n \sqrt{\delta^2-1})t} \right)$



DAMPING RATIO



- **Refer the first slide diagram for ref.**
- Depending on the value of damping ratio, second order system can be classified into:
 1. Undamped system, $\zeta = 0$
 2. Underdamped system, $0 < \zeta < 1$
 3. Critically damped system, $\zeta = 1$
 4. Overdamped system, $\zeta > 1$
- The characteristic equation is given by,

$$s^2 + 2\delta\omega_n s + \omega_n^2 = 0$$



DAMPING RATIO...

- The roots of characteristic equation is given by,

$$s = -\delta\omega_n \pm \omega_n \sqrt{\delta^2 - 1}$$

- The roots are imaginary when $\zeta = 0$
- The roots are real and equal when $\zeta = 1$
- The roots are real and unequal when $\zeta > 1$
- The roots are complex conjugate when $0 < \zeta < 1$



UNDAMPED SYSTEM



- Step Response of undamped second order system:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- When $\zeta = 0$,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$c(t) = (1 - \cos(\omega_n t))$$



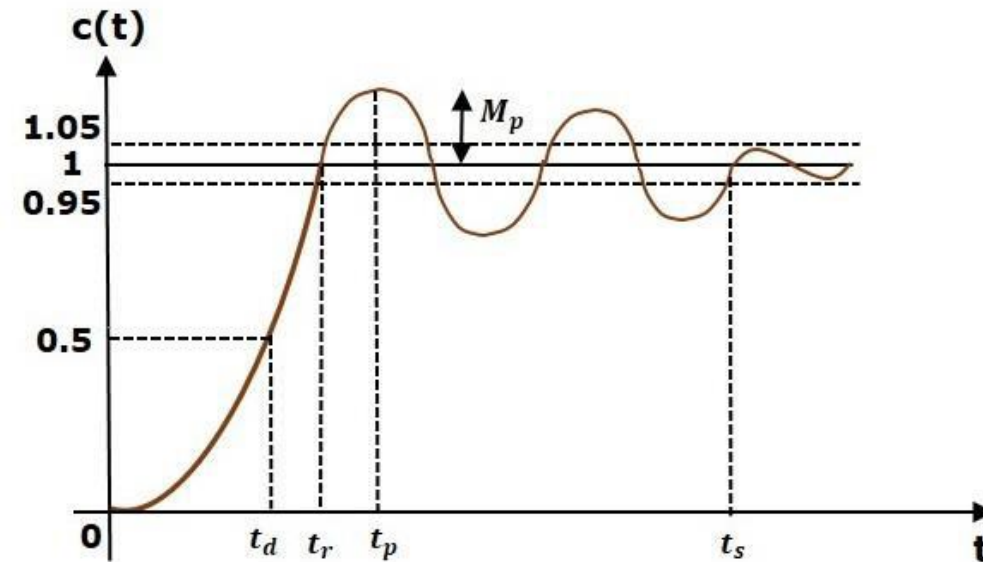
UNDERDAMPED SYSTEM

- Step response of a second order underdamped system:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{A}{s} + \frac{(Bs + C)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$c(t) = \left(1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1 - \delta^2}} \right) \sin(\omega_d t + \theta) \right)$$





CRITICALLY DAMPED SYSTEM



- Step response of a second order critically damped system:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

- When $\zeta = 1$,

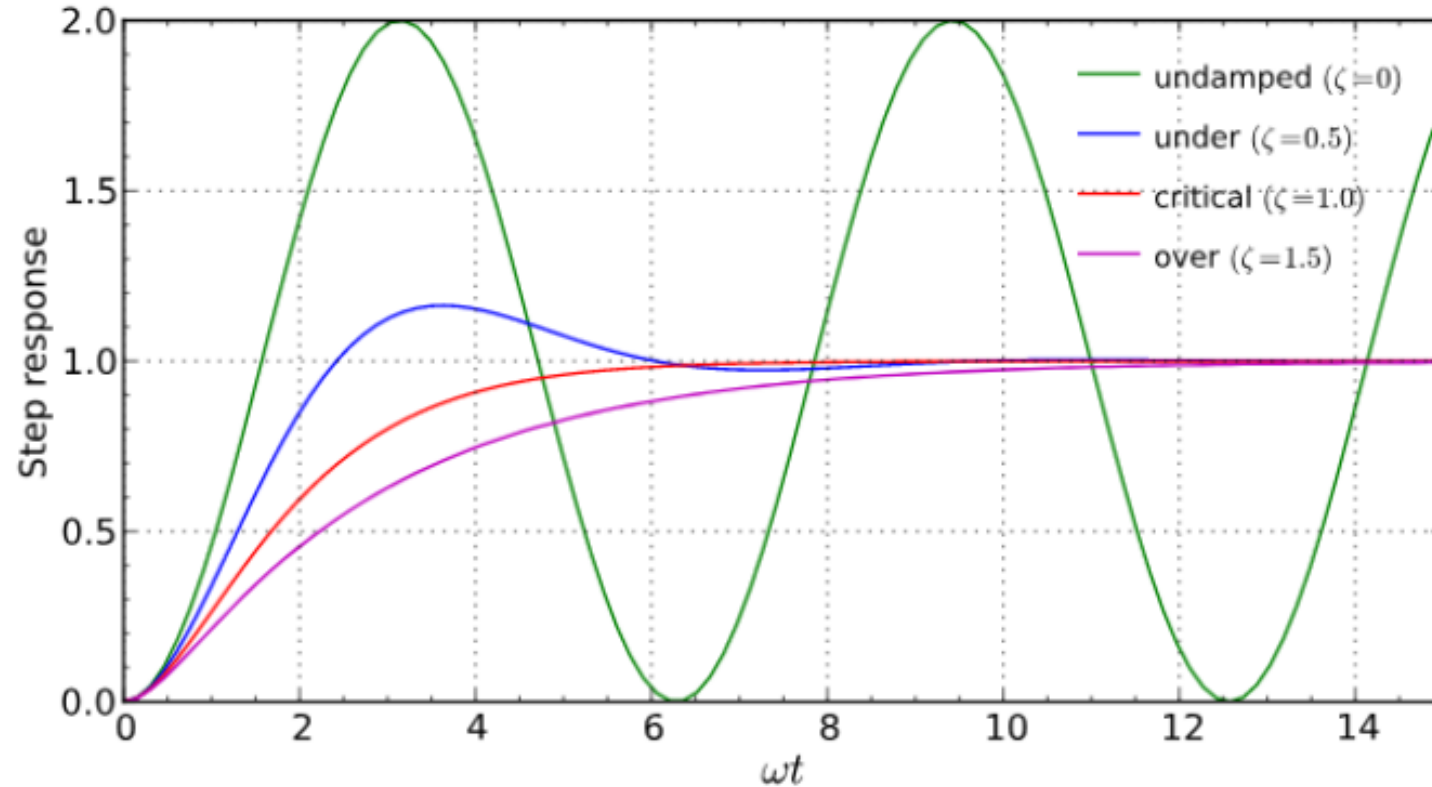
$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$C(s) = \frac{\omega_n^2}{s(s + \omega_n)^2} = \frac{A}{s} + \frac{B}{s + \omega_n} + \frac{C}{(s + \omega_n)^2}$$

$$c(t) = (1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t})$$



CRITICALLY DAMPED SYSTEM





SUMMARY

