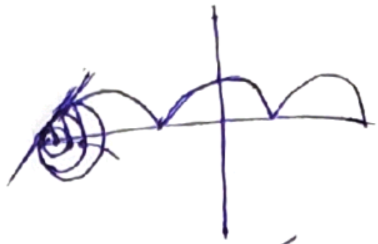
Evolute

The locus of centre of curvature of the given curve is called evolute. The given curve is called involute.



Locus: set of pts that satisfy certain condition

condition: pts in evolute curve is centre of circle

Curve	Cartesian eqn	Parametric eqn
Parabola	1. $y^2 = 4ax$ 2. $x^2 = 4ay$	1. $x = at^2$, $y = 2at$ 2. $x = 2at$, $y = at^2$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a\cos\theta$ $y = b\sin\theta$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x = a\sec\theta$ $y = b\tan\theta$
Rectangular hyperbola	$xy = c^2$	$x = ct$ $y = c/t$
Astroid	$x^{2/3} + y^{2/3} = a^{2/3}$	$x = a\cos^3\theta$, $y = a\sin^3\theta$.

Remarks

• y_1 in terms of t

i) $y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

ii) $y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$

1. Find the evolute of the parabola $y^2 = 4ax$

$x = at^2$ $\qquad$ $y = 2at$

$\frac{dx}{dt} = 2at$ $\qquad$ $\frac{dy}{dt} = 2a$

$$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{1}{t}\right)\left(\frac{1}{2at}\right)$$

$$= -\frac{1}{t^2}\left(\frac{1}{2at}\right)$$

$$= -\frac{1}{2at^3}$$

$$\bar{x} = x - y_1 \frac{(1+y_1^2)}{y_2}$$

$$= at^2 - \left(\frac{1}{t}\right)\frac{1+1/t^2}{-1/2at^3}$$

$$= at^2 + \frac{1}{\cancel{t}}\left(\frac{t^2+1}{\cancel{t^2}}\right)(2a\cancel{t^3})$$

$$= at^2 + (t^2+1)(2a)$$

$$= at^2 + 2at^2 + 2a$$

$$= 3at^2 + 2a \rightarrow (1)$$

$$\bar{Y} = y + \frac{1+y_1^2}{y_2}$$

$$= 2at + \frac{\left(1+\frac{1}{t^2}\right)}{-1/2at^3}$$

$$= 2at - \left(\frac{t^2+1}{\cancel{t^2}}\right)2a\cancel{t^3}$$

$$= 2at - 2at^3 - 2at$$

$$= -2at^3 \rightarrow (2)$$

We must remove t term in $\bar{x}, \bar{Y}$

$$\bar{x} = 3at^2 + 2a$$

$$3at^2 = \bar{x} - 2a$$

$$t^2 = \frac{\bar{x} - 2a}{3a}$$

$$\cancel{t = \frac{(\bar{x}-2a)^2}{9a^2}}$$

$$(t^2)^3 = \frac{(x-2a)^3}{(3a)^3}$$

$$t^6 = \frac{(\bar{x}-2a)^3}{27a^3} \rightarrow (3)$$

$$\bar{Y} = -2at^3$$

$$-2at^3 = \bar{Y}$$

$$t^3 = -\bar{Y}/2a$$

Squaring on both

$$(t^3)^2 = \frac{(-\bar{Y})^2}{(2a)^2}$$

$$= \frac{\bar{Y}^2}{4a^2} \rightarrow (4)$$

from (3) & (4)

$$\frac{(\bar{x}-2a)^3}{27a^3} = \frac{\bar{Y}^2}{4a^2}.$$

$$4(\bar{X}-2a)^3 = 27a\bar{Y}^2$$

The locus of $(\bar{x}, \bar{Y})$ is

$$4(x-2a)^3 = 27ay^2$$

2. Find the evolute of the ~~curve~~ parabola $x^2 = 4ay$

$x = 2at \qquad y = at^2$

$\frac{dx}{dt} = 2a \qquad \frac{dy}{dt} = 2at$

$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2at}{2a} = t$

$y_2 = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$

$= \frac{d}{dt}(t)\frac{dt}{dx}$

$= (1)\frac{1}{2a}$

$= 1/2a$

$\bar{x} = x - y_1\left(\frac{1+y_1^2}{y_2}\right)$

$= 2at - t\left(\frac{1+t^2}{1/2a}\right)$

$= 2at - (t+t^3)2a$

$= 2at - 2at - 2at^3$

$= -2at^3$

$\bar{y} = y + \frac{1+y_1^2}{y_2}$

$= at^2 + \left(\frac{1+t^2}{1/2a}\right)$

$= at^2 + 2a + 2at^2$

$= 3at^2 + 2a$

We must removing the t term

$-2at^3 = \bar{x}$

$t^3 = -\frac{\bar{x}}{2a}$

Squaring on both

$(t^3)^2 = \frac{(-\bar{x})^2}{(2a)^2}$

$t^6 = \frac{\bar{x}^2}{4a^2} \rightarrow (3)$

$3at^2 + 2a = \bar{y}$

$3at^2 = \bar{y} - 2a$

$t^2 = \frac{\bar{y}-2a}{3a}$

Cubing on both

$(t^2)^3 = \frac{(\bar{y}-2a)^3}{(3a)^3}$

$t^6 = \frac{(\bar{y}-2a)^3}{27a^3} \rightarrow (4)$

from (3) & (4)

$\frac{\bar{x}^2}{4a^2} = \frac{(\bar{y}-2a)^3}{27a^3}$

$27a\bar{x}^2 = 4(\bar{y}-2a)^3$

The locus of $(\bar{x}, \bar{y})$ is $27ax^2 = 4(y-2a)^3$

8. Find the evolute of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The parametric eqn is

$x = a\cos\theta \qquad y = b\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$

$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{b\cos\theta}{a\sin\theta}$

$d(u/v) = \frac{vu'-uv'}{v^2}$

$= -b/a \cot\theta$

$u = \cos\theta \qquad v = \sin\theta$

$u' = -\sin\theta \qquad v' = \cos\theta$

$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$\frac{d}{d\theta}\left(\frac{\cos\theta}{\sin\theta}\right) = \frac{\sin\theta(-\sin\theta) - \cos\theta\cos\theta}{\sin^2\theta}$

$= \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\left(\frac{d\theta}{dx}\right)$

$= -\frac{(\sin^2\theta+\cos^2\theta)}{\sin^2\theta}$

$= \frac{d}{d\theta}(-b/a\cot\theta)\frac{d\theta}{dx}$

$= \frac{-1}{\sin^2\theta}$

$= -b/a(-\operatorname{cosec}^2\theta)\frac{1}{-a\sin\theta}$

$= -\frac{b}{a^2\sin^3\theta}$

$\bar{x} = x - y_1\left(\frac{1+y_1^2}{y_2}\right)$

$= a\cos\theta + \frac{b}{a}\frac{\cos\theta}{\sin\theta}\,\frac{\left(1+\frac{b^2\cos^2\theta}{a^2\sin^2\theta}\right)}{-b/a^2\sin^3\theta}$

$= a\cos\theta + \frac{\cancel{b}}{a}\frac{\cos\theta}{\cancel{\sin\theta}}\;\frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^2\sin^2\theta}\;\frac{-a^2\sin^3\theta}{\cancel{b}}$

$= a\cos\theta - \frac{\cos\theta}{\cancel{a}}a^{\cancel{2}}\sin^2\theta - \frac{\cos\theta}{a}b^2\cos^2\theta$

$= a\cos\theta - a\cos\theta\sin^2\theta - b^2/a\cos^3\theta$

$= a\cos\theta(1-\sin^2\theta) - b^2/a\cos^3\theta$

$= a\cos^3\theta - b^2/a\cos^3\theta = \cos^3\theta(a - b^2/a)$

$\cos^3\theta = \frac{\bar{x}a}{a^2-b^2}$ Rising the power by 2/3

$\cos^2\theta = \frac{(xa)^{2/3}}{(a^2-b^2)^{2/3}}$

$$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$= b\sin\theta + \frac{1+\frac{b^2\cos^2\theta}{a^2\sin^2\theta}}{-b/a^2\sin^3\theta}$$

$$= b\sin\theta - \frac{a^2\sin^2\theta + b^2\cos^2\theta}{\cancel{a^2\sin^2\theta}} \quad \frac{\cancel{a^2}\sin^3\theta}{b}$$

$$= b\sin\theta - a^2/b\,\sin^3\theta - b\sin\theta\cos^2\theta$$

$$= b\sin\theta(1-\cos^2\theta) - a^2/b\,\sin^3\theta$$

$$= b\sin^3\theta - a^2/b\,\sin^3\theta$$

$$= \frac{b^2\sin^3\theta - a^2\sin^3\theta}{b}$$

$$= \frac{b^2-a^2}{b}\sin^3\theta$$

$$= -\left(\frac{a^2-b^2}{b}\right)\sin^3\theta$$

$$\sin^3\theta = \frac{-b\bar{x}}{a^2-b^2}$$

Raising the power by 2/3

$$\sin^2\theta = \frac{(+b\bar{x})^{2/3}}{(a^2-b^2)^{2/3}} \rightarrow (2)$$

(1)+(2) ⇒

$$\cos^2\theta + \sin^2\theta = \frac{(\bar{x}a)^{2/3} + (b\bar{y})^{2/3}}{(a^2-b^2)^{2/3}}$$

$$(\bar{x}a)^{2/3} + (\bar{y}b)^{2/3} = (a^2-b^2)^{2/3}$$

Locus of $(\bar{x}, \bar{y})$ is

$$(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$$

Find the evolute of hyperbola $x^2/a^2 - y^2/b^2 = 1$

Sol $x = a\sec\theta \qquad y = b\tan\theta$

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta \qquad \frac{dy}{d\theta} = b\sec^2\theta$$

$$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\sec^{\cancel{2}}\theta}{a\cancel{\sec\theta}\tan\theta}$$

$$= b/a \ \frac{1}{\cos\theta}\frac{\cos\theta}{\sin\theta} = b/a \ \operatorname{cosec}\theta$$

$$y_2 = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}(+b/a\operatorname{cosec}\theta)\frac{1}{a\sec\theta\tan\theta}$$

$$= -b/a(\operatorname{cosec}\theta\cot\theta)\frac{1}{a\sec\theta\tan\theta}$$

$$= -b/a\left(\frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta}\right)\frac{\cos\theta\cos\theta}{a\sin\theta}$$

$$= -b/a^2\frac{\cos^3\theta}{\sin^3\theta}$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a\sec\theta - \frac{(b/a\operatorname{cosec}\theta\left(1+\frac{b^2\operatorname{cosec}^2\theta}{a^2}\right)}{-b/a^2\frac{\cos 3\theta}{\sin^3\theta}}$$

$$= a\sec\theta + \frac{b}{a}\frac{1}{\sin\theta}\left(1+\frac{b^2}{a^2\sin^2\theta}\right)\frac{a^2\sin^3\theta}{b\cos 3\theta}$$

$$= a\sec\theta + \frac{\cancel{b}}{a\cancel{\sin\theta}}\left(\frac{a^2\sin^2\theta+b^2}{\cancel{a^2}\cancel{\sin^2\theta}}\right)\frac{\cancel{a^2}\cancel{\sin^3\theta}}{\cancel{b}\cos^3\theta}$$

$$= a\sec\theta + \frac{1}{a\cos^3\theta}(a^2\sin^2\theta+b^2)$$

$$= a\sec\theta + \frac{a\sin^2\theta}{\cos 3\theta} + \frac{b^2}{a}\sec^3\theta$$

$$= a\sec\theta + a\tan^2\theta\sec\theta + b^2/a\sec^3\theta$$

$$= a\sec\theta(1+\tan^2\theta) + b^2/a\sec^3\theta$$

$$= a\sec\theta\sec^2\theta + b^2/a\sec^3\theta = \frac{a^2+b^2}{a}\sec^3\theta$$

$$\sec^3\theta = \frac{a\bar{x}}{a^2+b^2}$$

Raising power by 2/3

$$\sec^2\theta = \frac{(a\bar{x})^{2/3}}{(a^2+b^2)^{2/3}} \rightarrow (1)$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= b\tan\theta + \frac{1+\frac{b^2}{a^2}\operatorname{cosec}^2\theta}{-b/a^2\,\frac{\cos 3\theta}{\sin 3\theta}}$$

$$= b\tan\theta - \left(1+\frac{b^2}{a^2\sin^2\theta}\right)\frac{a^2\sin^3\theta}{b\cos^3\theta}$$

$$= b\tan\theta - \left(\frac{a^2\sin^2\theta+b^2}{\cancel{a^2\sin^2\theta}}\right)\frac{\cancel{a^2}\sin^{\cancel{3}}\theta}{b\cos^3\theta}$$

$$= b\tan\theta - \frac{a^2\sin^3\theta}{b\cos^3\theta} - \frac{b^{\cancel{2}}\sin\theta}{\cancel{b}\cos^3\theta}$$

$$= b\tan\theta - a^2/b\,\tan^3\theta - b\tan\theta\sec^2\theta$$

$$= b\tan\theta\,(1-\sec^2\theta) - a^2/b\,\tan^3\theta$$

$$= b\tan\theta(-\tan^2\theta) - a^2/b\,\tan^3\theta$$

$$= -b\tan^3\theta - a^2/b\,\tan^3\theta$$

$$= \frac{-b^2-a^2}{b}\tan^3\theta$$

$$\bar{y} = -\tan^3\theta\left(\frac{a^2+b^2}{b}\right)$$

Raising the power by 2/3

$$\tan^2\theta = \frac{(-b\bar{y})^{2/3}}{(a^2+b^2)^{2/3}} = \frac{(b\bar{y})^{2/3}}{(a^2+b^2)^{2/3}} \rightarrow (2)$$

(1) - (2) $\Rightarrow$

$$\sec^2\theta - \tan^2\theta = \frac{(a\bar{x})^{2/3}}{(a^2+b^2)^{2/3}} - \frac{(b\bar{y})^{2/3}}{(a^2+b^2)^{2/3}}$$

$$1 = \frac{(a\bar{x})^{2/3} - (b\bar{y})^{2/3}}{(a^2+b^2)^{2/3}}$$

$$(a\bar{x})^{2/3} - (b\bar{y})^{2/3} = (a^2+b^2)^{2/3}$$

locus of $(\bar{x}, \bar{y})$ is

$$(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$$

ENVELOPE

The envelope of the family of the curve is the curve which touches each member of the family

Procedure to find to the envelope:

Method I : If the family of the curve is expressed as the quadratic form $Aa^2 + Ba + C = 0$, a is parameter the envelope is given by $B^2 - 4AC = 0$

Method II

1. Diff the given eqn w.r.to the parameter
2. Eliminate the parameter from the given curve