Centre $= (-g, -f)$
$= (-5/4, 1/2)$

$$r = \sqrt{g^2+f^2-c}$$
$$= \sqrt{\frac{25}{16} + \frac{1}{4} - 1/2}$$
$$= \sqrt{\frac{25+4-8}{16}}$$
$$= \sqrt{\frac{21}{16}}$$
$$= \frac{\sqrt{21}}{4}$$

Radius of curvature $\rho = \frac{\sqrt{21}}{4}$

Curvature $= \frac{1}{\rho}$
$$= \frac{4}{\sqrt{21}}$$

Example 3

Find the radius of curvature $xy = c^2$ at $x = c$

$$xy = c^2$$
$$y = \frac{c^2}{x}$$
$$\frac{dy}{dx} = c^2\left(-\frac{1}{x^2}\right)$$
$$= \frac{-c^2}{x^2}$$
$$\frac{dy}{dx}_{(c,c)} = \frac{-c^2}{c^2} = -1$$

Given $x = c$

$$\cancel{\boxed{y_1 = \frac{-c^2}{c^2} = -1}}$$

$$y = \frac{c^2}{c} = c$$

$$\frac{d^2y}{dx^2} = -c^2\left(\frac{-2}{x^3}\right)$$
$$= \frac{2c^2}{x^3}$$

$$\frac{d^2y}{dx^2}_{(c,c)} = \frac{2c^2}{c^3} = \frac{2}{c}$$

Radius of curvature $\rho = \dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$

$= \dfrac{\left[1+(-1)^2\right]^{3/2}}{2/c}$

$= \dfrac{(1+1)^{3/2}}{2/c}$

$= \dfrac{(\sqrt{2})^3}{2/c}$

$= \dfrac{\cancel{2}\sqrt{2}\,c}{\cancel{2}} = \sqrt{2}\,c$

4. Find the radius of curvature $y = e^x$ at $(0,1)$

Sol

given $y = e^x$

$\dfrac{dy}{dx} = e^x$, at $(0,1)$, $\left.\dfrac{dy}{dx}\right|_{(0,1)} = e^0 = 1$

$\dfrac{d^2y}{dx^2} = e^x$. at $(0,1)$ $\left.\dfrac{d^2y}{dx^2}\right|_{(0,1)} = e^0 = 1$

$\rho = \dfrac{(1+(y_1)^2)^{3/2}}{y_2}$

$= \dfrac{(1+1)^{3/2}}{1}$

$= 2^{3/2} = 2\sqrt{2}$

5. Find the radius of curvature $y = c\cosh\left(\frac{x}{c}\right)$ at $(0,c)$

$\dfrac{dy}{dx} = c\sinh\left(\frac{x}{c}\right)\frac{1}{c}$

$= \sinh\left(\frac{x}{c}\right)$

$\left.\dfrac{dy}{dx}\right|_{(0,c)} = \sinh\left(\frac{0}{c}\right)$

$= 0$

$$\frac{d^2y}{dx^2} = \cosh\left(\frac{x}{c}\right)\frac{1}{c}$$

$$= \cosh\left(\frac{0}{c}\right)\frac{1}{c}$$

$$= 1 \times \frac{1}{c} = \frac{1}{c}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+0)^{3/2}}{1/c}$$

$$\rho = c$$

6. Find the radius of curvature of the curve $\sqrt{x}+\sqrt{y}=1$ at $(1/4, 1/4)$

$$\sqrt{x}+\sqrt{y}=1$$

Diff w.r.to x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\, y_1 = 0 \rightarrow \cancel{(1)}$$

$$y_1 = -\frac{2\sqrt{y}}{2\sqrt{x}} = -\frac{\sqrt{y}}{\sqrt{x}} \rightarrow (1)$$

$$y_1(1/4, 1/4) = -\frac{\sqrt{1/4}}{\sqrt{1/4}}$$

$$= -1$$

Diff (1) w.r.to

$$d(uv) = \frac{vu' - uv'}{v^2} \qquad u = \sqrt{y}, \quad v = \sqrt{x}$$

$$u' = \frac{1}{2\sqrt{y}}\, y_1, \quad v' = \frac{1}{2\sqrt{x}}$$

$$y_2 = -\left[\frac{\sqrt{x}\frac{1}{2\sqrt{y}}\, y_1 - \sqrt{y}\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}\right]$$

$$y_2(1/4, 1/4) = -\left[\frac{\sqrt{1/4}\,\frac{1}{2\sqrt{1/4}}\,(-1) - \sqrt{(1/4)}\,\frac{1}{2\sqrt{1/4}}}{(1/4)}\right]$$

$$= -\left[\frac{-\frac{1}{2} - 1/2}{(1/4)}\right] = \frac{1/2 + 1/2}{1/4}$$

$$= 4$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+1)^{3/2}}{4}$$

$$= \frac{2^{3/2}}{4} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\underline{\rho = 1/\sqrt{2}}$$

* Find ρ where $x^3+y^3 = 3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Given $x^3+y^3 = 3axy$

Points $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Diff w.r. to x

$$\cancel{3}x^2 + \cancel{3}y^2 \cdot y_1 = \cancel{3}a\left[xy_1 + y\right]$$

$$y^2 \cdot y_1 = axy_1 + ay - x^2$$

$$y_1(y^2 - ax) = ay - x^2$$

$$y_1 = \frac{ay - x^2}{y^2 - ax} \rightarrow (1)$$

$$y_1\left(3a/2, 3a/2\right) = \frac{a\left(3a/2\right) - \frac{9a^2}{4}}{\frac{9a^2}{4} - a\left(\frac{3a}{2}\right)}$$

$$= \frac{3a^2/2 - 9a^2/4}{9a^2/4 - 3a^2/2}$$

$$= -\left[\frac{9a^2/4 - 3a^2/2}{9a^2/4 - 3a^2/2}\right]$$

$$= -1$$

Diff (1) w.r. to x

$$y_2 = \frac{vu' - uv'}{v^2}$$

$u = ay - x^2 \qquad v = y^2 - ax$

$u' = ay_1 - 2x \qquad v' = 2yy_1 - a$

$$y_2 = \frac{(y^2 - ax)(ay_1 - 2x) - (ay - x^2)(ay_1 - 2x)}{(y^2 - ax)^2}$$

$$= \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)\left(-a - \cancel{2}\frac{3a}{\cancel{2}}\right) - \left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)\left(-2\frac{3a}{2} - a\right)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$$

$$= \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)(-4a) + \left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)(-4a)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$$

$$= \frac{\cancel{(9a^2/4 - 3a^2/2)}(-4a - 4a)}{(9a^2/4 - 3a^2/2)^{\cancel{2}}}$$

$$= \frac{-8a}{\frac{9a^2 - 6a^2}{4}} = \frac{-32\cancel{a}}{3a^{\cancel{2}}}$$

$$= \frac{-32}{3a}$$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+1)^{3/2}}{-32/3a}$$

$$= \frac{(2)^{3/2}}{-32/3a} = \frac{2\sqrt{2} \times 3a}{-32}$$

$$\rho = -\frac{3a\sqrt{2}}{16}$$

$\rho = +3a\sqrt{2}/16$ (ρ is always +ve)

8 Find radius of curvature ρ, ~~at~~ $xy = 12$ at $(3, 4)$

$$xy = 12$$

$$y = \frac{12}{x}$$

Diff w.r.to x

$$y_1 = 12\left(-\frac{1}{x^2}\right)$$

$$= \frac{-12}{x^2} \rightarrow (1)$$

$$y_1(3,4) = -12/9$$

$$= -4/3$$

Diff (1) w.r.to x

$$y_2 = -12\left(\frac{-2}{x^3}\right)$$

$$= \frac{24}{x^3}$$

$$y_2(3,4) = \frac{24}{27}$$

$$= 8/9$$

The radius of curvature $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$\rho = \frac{\left[1+(-4/3)^2\right]^{3/2}}{8/9}$$

$$= \frac{(1+16/9)^{3/2}}{8/9} = \frac{(25/9)^{3/2}}{8/9}$$

$$= \frac{25\sqrt{25}}{9\sqrt{9}} \times \frac{9}{8}$$

$$= \frac{25 \times 5}{\cancel{9} \times 3} \times \frac{\cancel{9}}{8}$$

$$= \frac{125}{24}$$

9) Find ρ $x^4+y^4=2$ at $(1,1)$

Diff w.r.to x

$4x^3+4y^3y_1=0$

$y_1=-\frac{4x^3}{4y^3}$

$y_1=-x^3/y^3 \rightarrow (1)$

$y_1(1,1)=-1$

$u=x^3 \qquad v=y^3$

Diff (1) w.r.to x $\qquad u'=3x^2 \qquad v'=3y^2y_1$

$y_2=-\left[\frac{y^3(3x^2)-x^3(3y^2)y_1}{(y^3)^2}\right]$

$y_2(1,1)=-\left[\frac{1(3.1)-1(3.1.(-1))}{1}\right]$

$=-\left(\frac{3+3}{1}\right)$

$=-6$

$\rho=\frac{[1+y_1^2]^{3/2}}{y_2}$

$=\frac{(1+1)^{3/2}}{-6}$

$=-(2)^{3/2}/6 = -\frac{2\sqrt{2}}{6}$

$=-\sqrt{2}/3$

$\rho=\sqrt{2}/3$ (ρ is +ve)