Unit - 3

Centre and circle of curvature

The circle which touches the curve at point P and whose radius is equal to radius of curvature is known as circle of curvature. The centre of this circle is called centre of curvature. It is denoted by $(\bar{x}, \bar{y})$.

$$\bar{x} = x - y_1 \left(\frac{1+y_1^2}{y_2} \right)$$

$$\bar{y} = y + \left(\frac{1+y_1^2}{y_2} \right)$$

Circle of curvature $(x-\bar{x})(y-\bar{y}) = \rho$

1. Find the circle & circle of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $(a/4, a/4)$

Diff w.r.to x.

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} y_1 = 0$$

$$\frac{1}{2\sqrt{y}} y_1 = -\frac{1}{2\sqrt{x}}$$

$$y_1 = \frac{-\sqrt{y}}{\sqrt{x}} \rightarrow (1)$$

$$y_1(a/4, a/4) = \frac{-\sqrt{a/4}}{\sqrt{a/4}} = -1$$

Diff (1) w.r.to x

$$y_2 = -\left[\frac{\sqrt{x} \frac{1}{2\sqrt{y}} y_1 - \sqrt{y} \frac{1}{2\sqrt{x}}}{(\sqrt{x})^2} \right]$$

$$y_2(a/4, a/4) = -\left[\frac{\sqrt{a/4} \frac{1}{2\sqrt{a/4}} (-1) - \sqrt{a/4} \frac{1}{2\sqrt{a/4}}}{a/4} \right]$$

$$= -\left(\frac{-1/2 - 1/2}{a/4} \right) = \frac{1}{a/4}$$

$$= 4/a$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+1)^{3/2}}{4/a} = \frac{2\sqrt{2}\,a}{4}$$

$$= a/\sqrt{2}$$

$$\bar{x} = x - y_1\left(\frac{1+y_1^2}{y_2}\right), \quad \bar{y} = y + \left(\frac{1+y_1^2}{y_2}\right)$$

$$= a/4 - (-1)\left(\frac{1+1}{4/a}\right) \qquad = a/4 + \left(\frac{1+1}{4/a}\right)$$

$$= a/4 + a/2 \qquad \bar{y} = 3a/4$$

$$= 3a/4$$

Centre is $(\bar{x}, \bar{y}) = (3a/4, 3a/4)$

Circle of curvature is

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$(x - 3a/4)^2 + (y - 3a/4)^2 = a^2/2$$

2) Find circle of curvature $\sqrt{x}+\sqrt{y}=1$ at $(1/4, 1/4)$

Diff w.r. to x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} y_1 = 0$$

$$y_1 = -\frac{2\sqrt{y}}{2\sqrt{x}}$$

$$= -\sqrt{y}/\sqrt{x} \rightarrow (1)$$

$$y_1(1/4, 1/4) = -\sqrt{1/4}/\sqrt{1/4}$$

$$= -1$$

Diff (1) w.r. to x

$$y_2 = -\left[\frac{\sqrt{x}\,\frac{1}{2\sqrt{y}}\,y_1 - \sqrt{y}\,\frac{1}{2\sqrt{x}}}{x}\right]$$

$$y_2(1/4, 1/4) = -\left[\frac{\sqrt{1/4}\,\frac{1}{2\sqrt{1/4}}(-1) - \sqrt{1/4}\,/2\sqrt{1/4}}{1/4}\right]$$

$$= 4$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+1)^{3/2}}{4} = \frac{2\sqrt{2}}{4}$$

$$= \frac{1}{\sqrt{2}}$$

$$\bar{x} = x - y_1\left(\frac{1+y_1^2}{y_2}\right)$$

$$= 1/4 - (-1)\,\frac{1+1}{4}$$

$$= 3/4$$

$$\bar{y} = y + \left(\frac{1+y_1^2}{y_2}\right)$$

$$= 1/4 + 2/4 = 3/4$$

Circle of curvature is $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

$$(x-3/4)^2 + (y-3/4)^2 = 1/2$$

3. $xy = c^2$ at (c, c)

$$y = c^2/x$$

Diff w.r.to x

$$y_1 = c^2(-1/x^2) = -c^2/x^2 \quad \text{—(1)}$$

$$y_1(c,c) = -c^2/c^2 = -1$$

Diff (1) w.r.to x

$$y_2 = -c^2\left(\frac{-2}{x^3}\right)$$

$$= 2c^2/x^3$$

$$y_2(c,c) = 2c^2/c^3 = 2/c$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{4c} = \frac{2\sqrt{2}c}{2}$$

$$= \sqrt{2}c$$

$$\bar{x} = x - y_1\left(\frac{1+y_1^2}{y_2}\right), \quad \bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$= c - (-1)\left(\frac{1+1}{2/c}\right) \qquad = c + \left(\frac{1+1}{2/c}\right)$$

$$= c + 2/2 \times c \qquad = c + \frac{2c}{2}$$

$$= 2c \qquad = 2c$$

Centre is $(\bar{x}, \bar{y}) = (2c, 2c)$

Circle of curvature $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

$$(x-2c)^2 + (y-2c)^2 = 2c^2$$

4. $x^3 + y^3 = 3axy$ at $(3a/2, 3a/2)$

Diff w.r.to x

$$3x^2 + 3y^2 y_1 = 3a(xy_1 + y)$$

$$x^2 + y^2 y_1 = axy_1 + ay$$

$$(y^2 - ax)y_1 = ay - x^2$$

$$y_1 = \frac{ay - x^2}{y^2 - ax} \rightarrow (1)$$

$$y_1(3a/2, 3a/2) = \frac{a(3a/2) - (3a/2)^2}{(3a/2)^2 - a(3a/2)}$$

$$= -\left(\frac{9a^2/4 - 3a^2/2}{9a^2/4 - 3a^2/2}\right)$$

$$= -1$$

Diff (1) w.r.to x

$$y_1 = \frac{(y^2 - ax)(ay_1 - 2x) - (ay - x^2)(2yy_1 - a)}{(y^2 - ax)^2}$$

$$y_1(3a/2, 3a/2) = \frac{(3a/2)^2 - a(3a/2)\,(a(-1) - 2(3a/2)) - (a(3a/2) - (3a/2)^2)(2(3a/2)\text{[illegible]}}{[(3a/2)^2 - a(3a/2)]^2}$$

$$= \frac{(9a^2/4 - 3a^2/2)(-4a) - (3a^2/2 - 9a^2/4)(-4a)}{[9a^2/4 - 3a^2/2]^2}$$

$$= \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)(-8a)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$$

$$= \frac{-8a}{\frac{9a^2 - 6a^2}{4}} = \frac{-32}{3a} = -\frac{32}{3a}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{-32/3a} = \frac{3 \times 2\sqrt{2}\, a}{-32} = \frac{3\sqrt{2}\, a}{-16}$$

$$\rho = \frac{3\sqrt{2}\, a}{16} \quad \rho \text{ is +ve}$$

$$\bar{x} = x - y_1 \frac{(1+y_1^2)}{y_2} = 3a/2 - (-1)\frac{1+1}{-32/3a}$$

$$= 3a/2 + \frac{6a}{-32} = \frac{48a - 6a}{32} = \frac{42a}{32}$$

$$= 21a/16$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$= 3a/2 + \frac{(1+1)}{-32/3a} = \frac{48a - 6a}{32}$$

$$= 21a/16$$

The centre is $(\bar{x}, \bar{y}) = \left(21a/16, 21a/16\right)$

circle of curvature $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

$$\left(x - 21a/16\right)^2 + \left(y - 21a/16\right)^2 = \left(\frac{3\sqrt{2}\, a}{16}\right)^2$$

$$= \frac{9a^2}{128}$$

5. $y^2 = 12x$ at $(3, 6)$

Diff w.r.to x

$$2yy_1 = 12$$

$$y_1 = \frac{12}{2y}$$

$$= 6/y \rightarrow (1)$$

$$y_1(3,6) = 6/6 = 1$$

Diff w.r.to x

$$y_2 = 6 \frac{-1}{y^2} y_1$$

$$= -6/y^2$$

$$y_2(3,6) = -6/36$$

$$= -1/6$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+1)^{3/2}}{-1/6} = -\frac{2\sqrt{2}}{1} \times 6$$

$$= -12\sqrt{2}$$

$$\bar{x} = x - y_1 \frac{(1+y_1^2)}{y_2}$$

$$= 3 - 1 \frac{(1+1)}{-1/6}$$

$$= 3+12 = 15$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= 6 + \frac{(1+1)}{-1/6}$$

$$= 6 - 12 = -6$$

Circle of curvature

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$(x-15)^2 + (y+6)^2 = 288$$

6. $xy = 12$ at $(3, 4)$

$$y = 12/x$$

Diff w.r.to x

$$y_1 = -\frac{12}{x^2}$$

$$y_1(3,4) = \frac{-12}{9} = -4/3$$

Diff y_1 w.r.to x

$$y_2 = -12\left(\frac{-2}{x^3}\right)$$

$$= 24/x^3$$

$$y_2(3,4) = 24/27 = 8/9$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+16/9)^{3/2}}{8/9}$$

$$= \frac{(25/9)^{3/2}}{8} \times 9 = \frac{25\sqrt{25}}{9\sqrt{9}} \times \frac{9}{8}$$

~~$= \frac{25 \times 5 / 9 \times 3}{8} \times 9$~~ $= \frac{25 \times 5}{3 \times 8} = \frac{125}{24}$

$$\bar{x} = x - y_1 \frac{(1+y_1^2)}{y_2}$$

$$= 3 - (-4/3)\left(\frac{1+16/9}{8/9}\right)$$

$$= 3 + 4/3 \times \frac{9+16}{9} \times 9/8$$

$$= 3 + \frac{4 \times 25 \times 9}{3 \times 9 \times 8}$$

$$= \frac{18+25}{6} = 43/6$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$= 4 + \frac{1+16/9}{8/9}$$

$$= 4 + \frac{25}{9} \times \frac{9}{8}$$

$$= \frac{32+25}{8}$$

$$= 57/8$$

The centre of curvature $(\bar{x}, \bar{y})$

Circle of curvature is

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$