$$\tan^2\theta = \frac{(-b\bar{y})^{2/3}}{(a^2+b^2)^{2/3}} = \frac{(b\bar{y})^{2/3}}{(a^2+b^2)^{2/3}} \rightarrow (2)$$

$(1) - (2) \Rightarrow$

$$\sec^2\theta - \tan^2\theta = \frac{(a\bar{x})^{2/3}}{(a^2+b^2)^{2/3}} - \frac{(b\bar{y})^{2/3}}{(a^2+b^2)^{2/3}}$$

$$1 = \frac{(a\bar{x})^{2/3} - (b\bar{y})^{2/3}}{(a^2+b^2)^{2/3}}$$

$$(a\bar{x})^{2/3} - (b\bar{y})^{2/3} = (a^2+b^2)^{2/3}$$

locus of $(\bar{x}, \bar{y})$ is

$$(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$$

ENVELOPE

The envelope of the family of the curve is the curve which touches each member of the family

Procedure to find to the envelope:

Method I : If the family of the curve is expressed as the quadratic form $Aa^2 + Ba + C = 0$, a is parameter the envelope is given by $B^2 - 4AC = 0$

Method II

1. Diff the given eqn w.r.to the parameter
2. Eliminate the parameter from the given curve

Problems based on envelope with 1 parameter

1. Find the envelope of $y = mx + am^2$.

m is the parameter

Given $y = mx + am^2$.

Rearranging the eqn

$$am^2 + xm - y = 0$$

Comparing with general quadratic eqn

$$Aa^2 + Ba + C = 0 \quad \text{we get}$$

$$A = a$$

$$B = x$$

$$C = -y$$

The envelope is given by

$$B^2 - 4AC = 0$$

$$x^2 - 4a(-y) = 0$$

$$x^2 + 4ay = 0$$

2. $y = mx + \sqrt{a^2m^2 + b^2}$

$$y - mx = \sqrt{a^2m^2 + b^2}$$

$$(y - mx)^2 = a^2m^2 + b^2$$

$$y^2 - 2mxy + m^2x^2 - a^2m^2 - b^2 = 0$$

$$m^2(x^2 - a^2) - 2mxy + (y^2 - b^2) = 0$$

Comparing the general quadratic form

$Aa^2 + Ba + C = 0$

$A = x^2 - a^2$

$B = -2xy$

$C = y^2 - b^2$

The envelope is given by $B^2 - 4AC = 0$

$4x^2y^2 - 4(x^2 - a^2)(y^2 - b^2) = 0$

$x^2y^2 = (x^2 - a^2)(y^2 - b^2)$.

Exercise

3. Find the envelope of $(x-a)^2 + y^2 = 4a$

4. Find the envelope of $y = mx + \frac{a}{m}$

HM

5. $\frac{x^2}{\alpha} + \frac{y^2}{1-\alpha} = 1$, α is parameter.

6. Find the envelope of $x\cos\theta + y\sin\theta = a$, θ is the parameter.

$x\cos\theta + y\sin\theta = a \rightarrow (1)$

Diff (1) w.r.to θ

$-x\sin\theta + y\cos\theta = 0 \rightarrow (2)$

Squaring (1)

$(x\cos\theta + y\sin\theta)^2 = a^2$

$x^2\cos^2\theta + y^2\sin^2\theta + 2xy\cos\theta\sin\theta = a^2 \rightarrow (3)$

Squaring (2)

$(-x\sin\theta + y\cos)^2 = 0$

$x^2\sin^2\theta + y^2\cos^2\theta - 2xy\cos\theta\sin\theta = 0$

(3) + (4)

$x^2(\cos^2\theta + \sin^2\theta) + y^2(\sin^2\theta + \cos^2\theta) = a^2$

$x^2 + y^2 = a^2$ is the envelope

Problems based on envelop with 2 Parameter

1. Find the envelope of $\frac{x}{a}+\frac{y}{b}=1$, a & b are the 2 parameter recl to $a^n+b^n=c^n$, c is constant

$$\frac{x}{a}+\frac{y}{b}=1 \rightarrow (1)$$

$$a^n+b^n=c^n \rightarrow (2)$$

Diff (1) w.r.to a

$$\frac{-x}{a^2}+\frac{-y}{b^2}\frac{db}{da}=0$$

$$\frac{-y}{b^2}\frac{db}{da}=\frac{x}{a^2}$$

$$\frac{db}{da}=-\frac{xb^2}{ya^2} \rightarrow (3)$$

Diff (2) w.r.to a

$$na^{n-1}+nb^{n-1}\frac{db}{da}=0$$

$$\frac{db}{da}=\frac{-\cancel{n}a^{n-1}}{\cancel{n}b^{n-1}}$$

$$\frac{db}{da}=\frac{-a^{n-1}}{b^{n-1}} \rightarrow (4)$$

From (3) & (4)

$$\cancel{-}\frac{xb^2}{ya^2}=\frac{\cancel{-}a^{n-1}}{b^{n-1}}$$

$$\frac{x}{a^2a^{n-1}}=\frac{y}{b^2b^{n-1}}$$

$$\frac{x}{a^{n+1}}=\frac{y}{b^{n+1}} \Rightarrow \cancel{\frac{x}{a}\frac{1}{a^n}=\frac{y}{b}\frac{1}{b^n}}$$

$$\Rightarrow \frac{\frac{x}{a}}{a^n}=\frac{\frac{y}{b}}{b^n}=\frac{\frac{x}{a}+\frac{y}{b}}{a^n+b^n}=\frac{1}{c^n}$$

$$\frac{x}{a^{n+1}} = \frac{y}{b^{n+1}} = \frac{1}{c^n}$$

$$\frac{x}{a^{n+1}} = \frac{1}{c^n}$$

$$a^{n+1} = x c^n$$

Raising the power $\frac{n}{n+1}$

$$a^n = x^{n/n+1} c^{n^2/n+1}$$

$$\frac{y}{b^{n+1}} = \frac{1}{c^n}$$

$$b^{n+1} = y c^n$$

Raising the power $\frac{n}{n+1}$

$$b^n = y^{n/n+1} c^{n^2/n+1}$$

② $\Rightarrow$ $x^{n/n+1} c^{n^2/n+1} + y^{n/n+1} c^{n^2/n+1} = c^n$

$$x^{n/n+1} + y^{n/n+1} = \frac{c^n}{c^{n^2/n+1}}$$

$$x^{n/n+1} + y^{n/n+1} = c^{n/n+1}$$

2. $\frac{x}{a} + \frac{y}{b} = 1$, $a+b = c$

$$\frac{x}{a} + \frac{y}{b} = 1 \rightarrow ①$$

$$a + b = c \rightarrow ②$$

Diff ① w.r.to a

$$\frac{-x}{a^2} + \left(\frac{-y}{b^2}\right)\left(\frac{db}{da}\right) = 0$$

$$\frac{db}{da} = \frac{-x b^2}{y a^2} \rightarrow ③$$

Diff ② w.r.to a

$$1 + \frac{db}{da} = 0$$

$$\frac{db}{da} = -1 \rightarrow ④$$

From ③ & ④ $\frac{-xb^2}{ya^2} = -1 \Rightarrow xb^2 = ya^2$

$$\frac{x}{a^2} = \frac{y}{b^2}$$

$$\frac{\frac{x}{a}}{a} = \frac{\frac{y}{b}}{b} \Rightarrow \frac{\frac{x}{a} + \frac{y}{b}}{a+b}$$

$$\frac{x}{a^2} = \frac{y}{b^2} = \frac{1}{c}$$

$\frac{x}{a^2} = \frac{1}{c}$

$xc = a^2$

Raising the power by $1/2$

$a = (xc)^{1/2}$

$\frac{y}{b^2} = \frac{1}{c}$

$b^2 = yc$

Raising the power $1/2$

$b = (yc)^{1/2}$

(2) becomes

$$(xc)^{1/2} + (yc)^{1/2} = c$$

$$c^{1/2} (x^{1/2} + y^{1/2}) = c$$

$$x^{1/2} + y^{1/2} = c^{1/2}$$