13

CHAPTER

DEFLECTION OF CANTILEVERS

ABC is fix

13.1. INTRODUCTION

Cantilever is a beam whose one end is fixed and other end is free. In this chapter we shall discuss the methods of finding slope and deflection for the cantilevers when they are subjected to various types of loading. The important methods are (*i*) Double integration method (*ii*) Macaulay's method and (*iii*) Moment-area-method. These methods have also been used for finding deflections and slope of the simply supported beams.

13.2. DEFLECTION OF A CANTILEVER WITH A POINT LOAD AT THE FREE END BY DOUBLE INTEGRATION METHOD

A cantilever *AB* of length *L* fixed at the point *A* and free at the point *B* and carrying a point load at the free end *B* is shown in Fig. 13.1. *AB* shows the position of cantilever before any load is applied whereas *AB′* shows the position of the cantilever after loading.

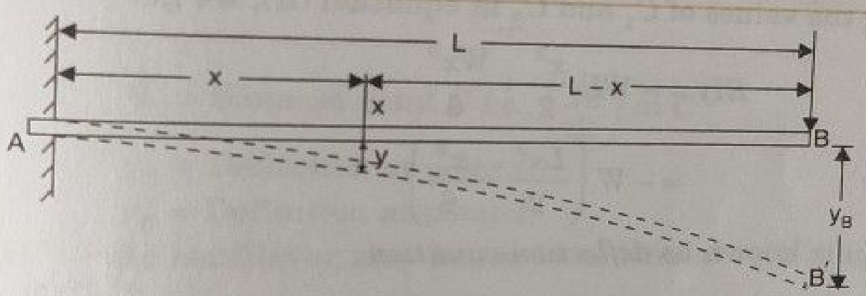

Fig. 13.1

Consider a section *X*, at a distance *x* from the fixed end *A*. The B.M. at this section is given by,

$$M_x = -W(L-x) \qquad \text{(Minus sign due to hogging)}$$

But B.M. at any section is also given by equation (12.3) as

$$M = EI\frac{d^2y}{dx^2}$$

Equating the two values of B.M., we get

$$EI\frac{d^2y}{dx^2} = -W(L-x) = -WL + W.x$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = -WLx + \frac{Wx^2}{2} + C_1 \qquad ...(i)$$

Integrating again, we get

$$EIy = -WL\frac{x^2}{2} + \frac{W}{2}\frac{x^3}{3} + C_1x + C_2 \quad ...(ii)$$

where C_1 and C_2 are constant of integrations. Their values are obtained from boundary conditions, which are : (*i*) at $x = 0, y = 0$ (*ii*) $x = 0, \frac{dy}{dx} = 0$

[At the fixed end, deflection and slopes are zero]

(*i*) By substituting $x = 0, y = 0$ in equation (*ii*), we get

$$0 = 0 + 0 + 0 + C_2 \quad \therefore \quad C_2 = 0$$

(*ii*) By substituting $x = 0, \frac{dy}{dx} = 0$ in equation (*i*), we get

$$0 = 0 + 0 + C_1 \quad \therefore \quad C_1 = 0$$

Substituting the value of C_1 in equation (*i*), we get

$$EI\frac{dy}{dx} = -WLx + \frac{Wx^2}{2}$$

$$= -W\left(Lx - \frac{x^2}{2}\right) \quad ...(iii)$$

Equation (*iii*) is known as *slope equation*. We can find the slope at any point on the cantilever by substituting the value of *x*. The slope and deflection are maximum at the free end. These can be determined by substituting $x = L$ in these equations.

Substituting the values of C_1 and C_2 in equation (*ii*), we get

$$EIy = -WL\frac{x^2}{2} + \frac{Wx^3}{6} \quad (\because C_1 = 0, C_2 = 0)$$

$$= -W\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) \quad ...(iv)$$

Equation (*iv*) is known as *deflection equation*.

Let θ_B = slope at the free end *B* *i.e.*, $\left(\frac{dy}{dx}\right)$ at $B = \theta_B$ and

y_B = Deflection at the free end *B*

(*a*) Substituting θ_B for $\frac{dy}{dx}$ and $x = L$ in equation (*iii*), we get

$$EI.\theta_B = -W\left(L.L - \frac{L^2}{2}\right) = -W.\frac{L^2}{2}$$

$$\therefore \quad \theta_B = -\frac{WL^2}{2EI} \quad ...(13.1)$$

Negative sign shows that tangent at *B* makes an angle in the anti-clockwise direction with *AB*

$$\therefore \quad \theta_B = \frac{WL^2}{2EI} \quad ...(13.1A)$$

(b) Substituting y_B for y and $x = L$ in equation (iv), we get

$$EI.y_B = -W\left(L.\frac{L^2}{2} - \frac{L^3}{6}\right) = -W\left(\frac{L^3}{2} - \frac{L^3}{6}\right) = -W.\frac{L^3}{3}$$

$\therefore$ $$y_B = -\frac{WL^3}{3EI} \quad ...(13.2)$$

(Negative sign shows that deflection is downwards)

$\therefore$ Downward deflection, $$y_B = \frac{WL^3}{3EI} \quad ...(13.2\ A)$$

13.3. DEFLECTION OF A CANTILEVER WITH A POINT LOAD AT A DISTANCE 'a' FROM THE FIXED END

A cantilever AB of length L fixed at point A and free at point B and carrying a point load W at a distance 'a' from the fixed end A, is shown in Fig. 13.2.

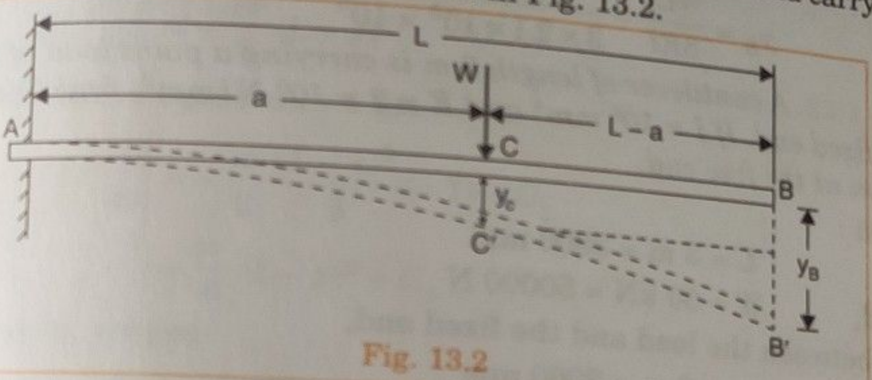

Fig. 13.2

Let θ_C = Slope at point C i.e., $\left(\frac{dy}{dx}\right)$ at C

y_C = Deflection at point C

y_B = Deflection at point B

The portion AC of the cantilever may be taken as similar to a cantilever in Art. 13.1 (i.e., load at the free end).

$\therefore$ $$\theta_C = +\frac{Wa^2}{2EI} \quad \text{[In equation (13.1 A) change } L \text{ to } a\text{]}$$

and $$y_C = \frac{Wa^3}{3EI} \quad \text{[In equation (13.2 A) change } L \text{ to } a\text{]}$$

The beam will bend only between A and C, but from C to B it will remain straight since B.M. between C and B is zero.

Since the portion CB of the cantilever is straight, therefore

Slope at C = slope at B

or $$\theta_C = \theta_B = \frac{Wa^2}{2EI} \quad ...(13.3)$$

Now from Fig. 13.2, we have

$$y_B = y_C + \theta_C(L-a)$$

$$= \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(L-a) \quad \left(\because \theta_C = \frac{Wa^2}{2EI}\right) \quad ...(13.4)$$

13.4. DEFLECTION OF A CANTILEVER WITH A UNIFORMLY DISTRIBUTED LOAD

A cantilever AB of length L fixed at the point A and free at the point B and carrying a uniformly distributed load of w per unit length over the whole length, is shown in Fig. 13.3.

Consider a section X, at a distance x from the fixed end A. The B.M. at this section is given by,

$$M_x = -w(L-x) \cdot \frac{(L-x)}{2} \qquad \text{(Minus sign due to hogging)}$$

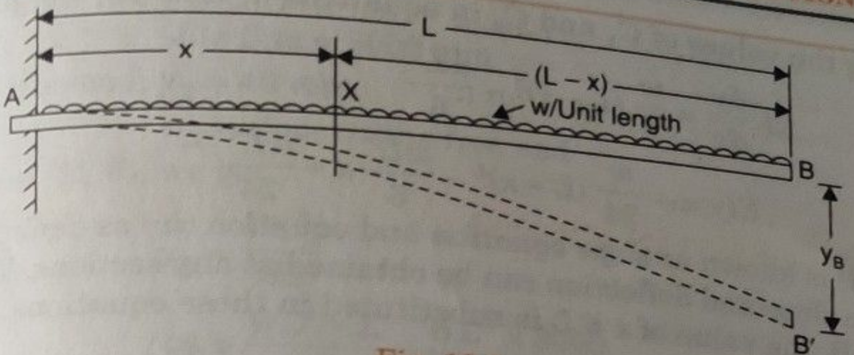

Fig. 13.3

But B.M. at any section is also given by equation (12.3) as

$$M = EI\frac{d^2y}{dx^2}$$

Equating the two values of B.M., we get

$$EI\frac{d^2y}{dx^2} = -\frac{w}{2}(L-x)^2$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = -\frac{w}{2}\frac{(L-x)^3}{3}(-1) + C_1$$

$$= \frac{w}{6}(L-x)^3 + C_1 \qquad \text{...(i)}$$

Integrating again, we get

$$EIy = \frac{w}{6}\cdot\frac{(L-x)^4}{4}(-1) + C_1x + C_2$$

$$= -\frac{w}{24}(L-x)^4 + C_1x + C_2 \qquad \text{...(ii)}$$

where C_1 and C_2 are constant of integrations. Their values are obtained from boundary conditions, which are : (*i*) at $x = 0, y = 0$ and (*ii*) at $x = 0, \frac{dy}{dx} = 0$ (as the deflection and slope at fixed end A are zero).

(*i*) By substituting $x = 0, y = 0$ in equation (*ii*), we get

$$0 = -\frac{w}{24}(L-0)^4 + C_1 \times 0 + C_2 = -\frac{wL^4}{24} + C_2$$

$$\therefore \quad C_2 = \frac{wL^4}{24}$$

(*ii*) By substituting $x = 0$ and $\frac{dy}{dx} = 0$ in equation (*i*), we get

$$0 = \frac{w}{6}(L-0)^3 + C_1 = \frac{wL^3}{6} + C_1$$

$$\therefore \quad C_1 = -\frac{wL^3}{6}$$

Substituting the values of C_1 and C_2 in equations (*i*) and (*ii*), we get

$$EI\frac{dy}{dx} = \frac{w}{6}(L-x)^3 - \frac{wL^3}{6} \quad ...(iii)$$

and

$$EIy = -\frac{w}{24}(L-x)^4 - \frac{wL^3}{6}x + \frac{wL^4}{24} \quad ...(iv)$$

Equation (*iii*) is known as *slope equation* and equation (*iv*) as *deflection equation*. From these equations the slope and deflection can be obtained at any sections. To find the slope and deflection at point *B*, the value of $x = L$ is substituted in these equations.

Let θ_B = Slope at the free end *B i.e.*, $\left(\frac{dy}{dx}\right)$ at *B*

y_B = Deflection at the free end *B*.

From equation (*iii*), we get slope at *B* as

$$EI.\theta_B = \frac{w}{6}(L-L)^3 - \frac{wL^3}{6} = -\frac{wL^3}{6}$$

$$\therefore \quad \theta_B = -\frac{wL^3}{6EI} = -\frac{WL^2}{6EI} \quad (\because \; W = \text{Total load} = w.L) \quad ...(13.5)$$

From equation (*iv*), we get the deflection at *B* as

$$EI.y_B = -\frac{w}{24}(L-L)^4 - \frac{wL^3}{6}\times L + \frac{wL^4}{24}$$

$$= -\frac{wL^4}{6} + \frac{wL^4}{24} = -\frac{3}{24}wL^4 = -\frac{wL^4}{8}$$

$$\therefore \quad y_B = -\frac{wL^4}{8EI} = -\frac{WL^3}{8EI} \quad (\because \; W = w.L)$$

$\therefore$ Downward deflection at *B*,

$$y_B = \frac{wL^4}{8EI} = \frac{WL^3}{8EI} \quad ...(13.6)$$

Problem 13.3. *A cantilever of length 2.5 m carries a uniformly distributed load of 16.4 kN per metre length over the entire length. If the moment of inertia of the beam = 7.95 × 10^7 mm^4 and value of E = 2 × 10^5 N/mm^2, determine the deflection at the free end.*

Sol. Given :

Length, $L = 2.5 \text{ m} = 2500 \text{ mm}$

U.d.l., $w = 16.4 \text{ kN/m}$

$\therefore$ Total load, $W = w \times L = 16.4 \times 2.5 = 41 \text{ kN} = 41000 \text{ N}$

Value of $I = 7.95 \times 10^7 \text{ mm}^4$

Value of $E = 2 \times 10^5 \text{ N/mm}^2$

Let y_B = Deflection at the free end,

Using equation (13.6), we get

$$y_B = \frac{WL^3}{8EI} = \frac{41000 \times 2500^3}{8 \times 2 \times 10^5 \times 7.95 \times 10^7}$$

$= \mathbf{5.036 \text{ mm}}$. **Ans.**

Problem 13.4. *A cantilever of length 3 m carries a uniformly distributed load over the entire length. If the deflection at the free end is 40 mm, find the slope at the free end.*

$$w = \frac{5 \times 8 \times 2 \times 10^{5} \times 8 \times 10^{\dots}}{2.5 \times 2500^3} = 16384 \text{ N/m}$$

or $= \mathbf{16.384 \ kN/m.}$ **Ans.**

13.5. DEFLECTION OF A CANTILEVER WITH A UNIFORMLY DISTRIBUTED LOAD FOR A DISTANCE 'a' FROM THE FIXED END

A cantilever AB of length L fixed at the point A and free at the point B and carrying a uniformly distributed load of w/m length for a distance 'a' from the fixed end, is shown in Fig. 13.4.

The beam will bend only between A and C, but from C to B it will remain straight since B.M. between C and B is zero. The deflected shape of the cantilever is shown by $AC'B'$ in which portion $C'B'$ is straight.

Let θ_C = Slope at C, *i.e.*, $\left(\frac{dy}{dx}\right)$ at C

y_C = Deflection at point C, and

y_B = Deflection at point B.

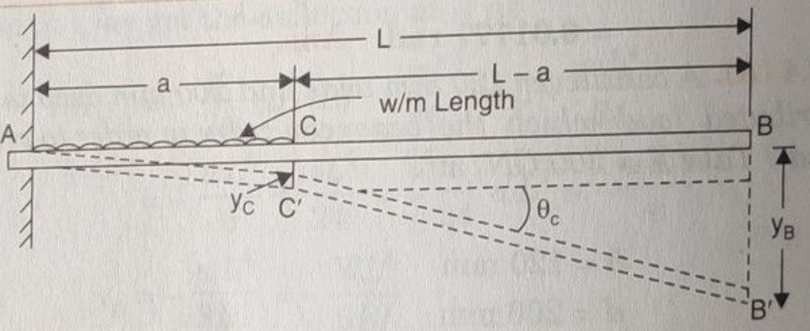

Fig. 13.4

The portion AC of the cantilever may be taken as similar to a cantilever in Art. 13.4.

$$\therefore \quad \theta_C = \frac{w \cdot a^3}{6EI} \qquad \text{[In equation (13.5) put } L = a]$$

and

$$y_C = \frac{w \cdot a^4}{8EI} \qquad \text{[In equation (13.6) put } L = a]$$

Since the portion $C'B'$ of the cantilever is straight, therefore slope at C = slope at B

or

$$\theta_C = \theta_B = \frac{wa^3}{6EI} \qquad \text{...(13.7)}$$

Now from Fig. 13.4, we have

$$y_B = y_C + \theta_C (L - a)$$

$$= \frac{wa^4}{8EI} + \frac{w \cdot a^3}{6EI}(L - a) \qquad \text{...(13.8)}$$

13.6. DEFLECTION OF A CANTILEVER WITH A UNIFORMLY DISTRIBUTED LOAD FOR A DISTANCE 'a' FROM THE FREE END

A cantilever AB of length L fixed at the point A and free at the point B and carrying a uniformly distributed load of w/m length for a distance 'a' from the free end is shown in Fig. 13.5 (a).

The slope and deflection at the point B is determined by considering :

(i) the whole cantilever AB loaded with a uniformly distributed load of w per unit length as shown in Fig. 13.5 (b).

(ii) a part of cantilever from A to C of length $(L - a)$ loaded with an upward uniformly distributed load of w per unit length as shown in Fig. 13.5 (c).

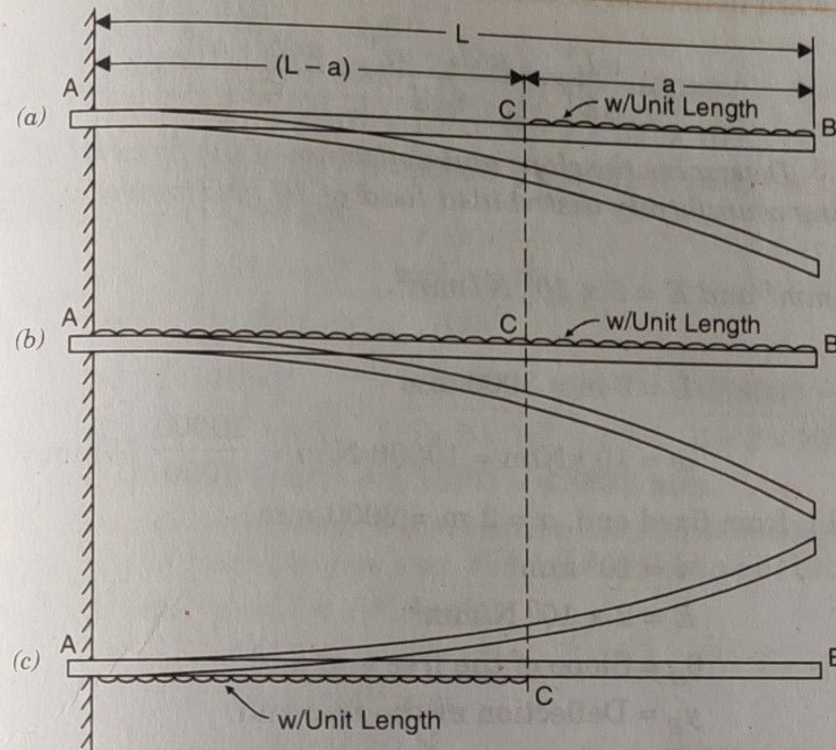

Fig. 13.5

Then slope at B = Slope due to downward uniform load over the whole length – slope due to upward uniform load from A to C

and deflection at B = Deflection due to downward uniform load over the whole length – deflection due to upward uniform load from A to C.

(a) Now slope at B due to downward uniformly distributed load over the whole length

$$= \frac{wL^3}{6EI}$$

(b) Slope at B or at C due to upward uniformly distributed load over the length $(L - a)$

$$= \frac{w(L-a)^3}{6EI}$$

Hence net slope at B is given by,

$$\theta_B = \frac{wL^3}{6EI} - \frac{w(L-a)^3}{6EI} \quad \text{...(13.9)}$$

The downward deflection of point B due to downward distributed load over the whole length AB

$$= \frac{wL^4}{8EI}$$

The upward deflection of point B due to upward uniformly distributed load acting on the portion AC = upward deflection of C + slope at $C \times CB$

$$= \frac{w(L-a)^4}{8EI} + \frac{w.(L-a)^3}{6EI} \times a \qquad (\because \; CB = a)$$

$\therefore$ Net downward deflection of the free end B is given by

$$y_B = \frac{wL^4}{8EI} - \left[\frac{w(L-a)^4}{8EI} + \frac{w(L-a)^3}{6EI} \times a\right] \qquad ...(13.10)$$