Castigliano's theorem:

→ Castigliano's first theorem

→ Castigliano's Second theorem.

1) <u>Castigliano's first theorem</u>:

In the linearly elastic system the partial derivative of the total strain Energy stored in the structure with respect to a load gives displacement at that Point in the direction of Load.

$$\boxed{\Delta_i = \frac{\partial U}{\partial P_i}}$$

where,

$U \rightarrow$ Total Strain Energy.

$P_i = M_f$ = Loads.

$\Delta_i = \theta_f$ = Deflection Displacement

The load may be force (or) Moment

2) <u>Castigliano's Second theorem</u>.

In the linearly Elastic System, the partial derivative of the total strain Energy stored in the Structure with respect to the displacement at a Point is equal to the force at that Point.

$$\boxed{P = \frac{\partial U}{\partial \delta}}$$

## Uses of Castigliano theorem:

→ To determine the displacements of Complicated Structures.

→ To find the deflection of beams due to shearing or bending if the total strain Energy due to shearing forces or bending moments is known.

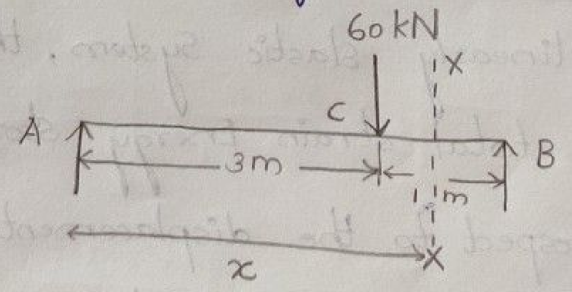
→ To find the deflections of Curved beams, Springs etc.

To find out deflections or Rotations at a point of structure where there is no load,

$$x = \left(\frac{\partial U}{\partial W}\right)_{W=0} \quad \& \quad \phi = \left(\frac{\partial U}{\partial M}\right)_{M=0}$$

1) Using Castigliano's theorem, Obtain the deflection under a single concentrated load applied to a Simply supported beam shown in fig. $EI = 2.2\ MN/m^2$.

Let the load at C be denoted by W. Taking moments about B, we get




$$R_A \times 4 = W \times 1$$

$$R_A = \frac{W}{4}$$

Consider a Section XX at a distance $x$ from A.

$$M_x = R_A \times x - W(x-3)$$

$$= \frac{W}{4} \times x - W(x-3)$$

$$= \frac{Wx}{4} - W(x-3)$$

$$\frac{\partial M}{\partial W} = \frac{x}{4} - (x-3)$$

Now, $\delta = \int \frac{M}{EI} \cdot \frac{\partial M}{\partial W}\, dx$

$$= \frac{1}{EI}\int_0^3 \frac{Wx}{4} \times \frac{x}{4}\, dx + \frac{1}{EI}\int_3^4 \left\{\frac{Wx}{4} - W(x-3)\right\} \times \left\{\frac{x}{4} - (x-3)\right\} dx$$

$$= \frac{W}{16\,EI}\int_0^3 x^2 \cdot dx + \frac{W}{EI}\int_3^4 \left\{\left(\frac{x}{4} - (x-3)\right)\right\}^2 dx$$

$$= \frac{W}{16\,EI}\int_0^3 x^2 \cdot dx + \frac{W}{EI}\int_3^4 \left(\frac{x - 4x + 12}{4}\right)^2 dx$$

$$= \frac{W}{16EI}\int_0^3 x^2 \cdot dx + \frac{W}{16\,EI}\int_3^4 (x - 4x + 12)^2\, dx$$

$$= \frac{W}{16EI}\int_0^3 x^2 \cdot dx + \frac{9W}{16\,EI}\int_3^4 (x^2 - 8x + 16)\, dx$$

$$= \frac{W}{16\,EI}\left[\frac{x^3}{3}\right]_0^3 + \frac{9W}{16EI}\left[\frac{x^3}{3} - \frac{8x^2}{2} + 16x\right]_3^4$$

$$= \frac{W}{16EI}\left[\frac{\overset{9}{\cancel{27}}}{\cancel{3}}\right] + \frac{9W}{16EI}\left[\frac{64}{3} - \frac{\cancel{1}28}{2} + \cancel{6}4 - \frac{\overset{9}{\cancel{27}}}{\cancel{3}} + \frac{\overset{36}{\cancel{72}}}{2} - 48\right]$$

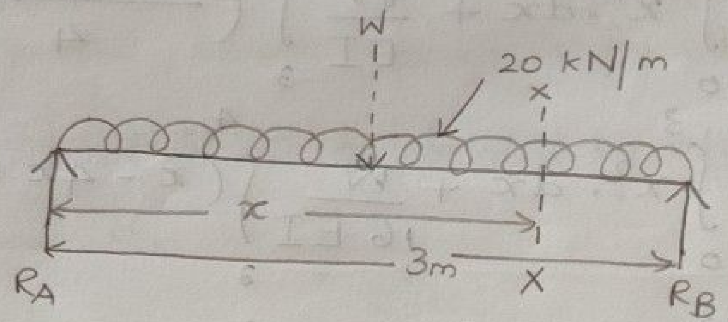
$$= \frac{9W}{16EI} + \frac{9W}{16EI}\left[0.333\right]$$

$$= \frac{9W}{16EI}\left[1 + 0.333\right]$$

$$= \frac{0.75\,W}{EI}$$

$$\delta = \frac{0.75 \times 60 \times 10^3}{2.2 \times 10^6} = 0.02045 \text{ m}$$
$$\text{(or) } 20.45 \text{ mm}$$

2) A Beam Simply Supported over a Span of 3m carries a UDL of 20 kN/m over Entire span. Taking EI = 2.25 MNm² & Using Castigliano's theorem, determine the deflection at Centre of Beam.




$l = 3$ m

$w = 20$ kN/m.

$EI = 2.25$ MNm².

Deflection at the Centre $\delta$:

Let W (kN) be the dummy Load at Mid Span.

Taking moments about B, We get

$$R_A \times 3 = 20 \times 3 \times \frac{3}{2} + W \times 1.5$$

$$= \frac{60 \times 3}{2} + 1.5W$$

$$R_A = \frac{90 + 1.5W}{3}$$

$$R_A = (30 + 0.5W)\ \text{kN}.$$

Consider a Section XX at a distance $x$ from A.

Then,

$$M_x = (30 + 0.5W)x - 20 \times x \times \frac{x}{2} - W(x - 1.5)$$

$$\frac{\partial M_x}{\partial W} = 0.5x - (x - 1.5)$$

Deflection $\delta = \int \frac{M}{EI} \cdot \frac{\partial M}{\partial W}\, dx$

$$= \frac{1}{EI}\int_0^{1.5} (30 + 0.5W)x - 10x^2) \times 0.5x\, dx$$

$$+ \frac{1}{EI}\int_{1.5}^{3} \{(30 + 0.5W)x - 10x^2 - W(x - 1.5)\} \times \{0.5x - (x - 1.5)\}\, dx$$

$$= \frac{1}{EI}\int_0^{1.5} \{(15 + 0.25W)x^2 - 5x^3\}\, dx$$

$$+ \frac{1}{EI}\int_{1.5}^{3} \{(30x + 0.5Wx - 10x^2 - Wx + 1.5W)(0.5x - x + 1.5)\, dx$$

$$= \frac{1}{EI}\int_0^{1.5} \{(15 + 0.25W)x^2 - 5x^3\}\, dx$$

$$+ \frac{1}{EI}\int_{1.5}^{3} \{(30 - 0.5W)x - 10x^2 + 1.5W\}(1.5 - 0.5x)\, dx$$

$$= \frac{1}{EI}\int_0^{1.5}\left\{(15+0.25W)x^2 - 5x^3\right\}dx$$

$$+ \frac{1}{EI}\int_{1.5}^{3}\left\{(45-0.75W)x - 15x^2 + 2.25W - (15-0.25W)x^2 + 5x^3 - 0.75Wx\right\}dx$$

$$= \frac{1}{EI}\int_0^{1.5}\left\{(15+0.25W)x^2 - 5x^3\right\}dx$$

$$+ \frac{1}{EI}\int_{1.5}^{3}\left\{(45-1.5W)x + ((0.25W)-30)x^2 + 5x^3 + 2.25W\right\}dx$$

$$= \frac{1}{EI}\left[(15+0.25W)\frac{x^3}{3} - \frac{5}{4}x^4\right]_0^{1.5}$$

$$+ \frac{1}{EI}\left[(45-1.5W)\frac{x^2}{2} + (0.25W-30)\frac{x^3}{3} + \frac{5}{4}x^4 + 2.25Wx\right]_{1.5}^{3}$$

Put $W=0$, We get,

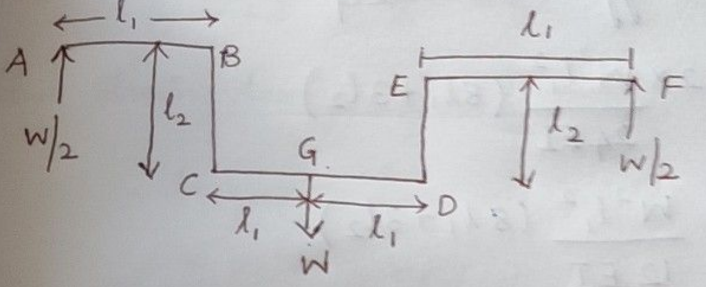
$$\delta = \frac{1}{EI}\left[5\times1.5^3 - 1.25\times1.5^4\right] + \frac{1}{EI}\left[22.5\times6.75 - 10\times23.625 + 1.25\times75.94\right]$$

$$= \frac{1}{EI}\left[16.875 - 6.33\right] + \frac{1}{EI}(151.88 - 236.25 + 94.92)$$

$$= \frac{21.09}{EI} = \frac{21.09\times10^3}{2.25\times10^6} = 9.37\times10^{-3}$$

$$\boxed{\delta = 9.37\ mm}$$

3) Using Castigliano's theorem find the Central deflection of the uniform bend shown in fig.




strain Energy for the portion ABCG,

$$U_{ABCG} = \int_0^{l_1} \frac{\left(\frac{W}{2}x\right)^2}{2EI} + \int_0^{l_2} \frac{\left(\frac{W}{2}l_1\right)^2}{2EI} + \int_0^{l_1} \frac{\left(\frac{W}{2}(x+l_1)\right)^2 dx}{2EI}$$

$$= \int_0^{l_1} \frac{\frac{W^2}{4}x^2}{2EI} + \int_0^{l_2} \frac{\frac{W^2}{4}l_1^2}{2EI} + \int_0^{l_1} \frac{\frac{W^2}{4}(x+l_1)^2}{2EI} dx$$

$$= \frac{W^2}{8EI}\int_0^{l_1} x^2 dx + \int_0^{l_2} \frac{W^2 l_1^2}{8EI} + \int_0^{l_1} \frac{W^2}{8EI}(x+l_1)^2 dx$$

$$= \frac{W^2}{8EI}\left[\frac{l_1^3}{3}\right] + \int_0^{l_2} \frac{W^2 l_1^2}{8EI} + \frac{W^2}{8EI}\left[\frac{(x+l_1)^3}{3}\right]_0^{l_1}$$

$$= \frac{W^2 l_1^3}{24EI} + \frac{W^2 l_1^2}{8EI}[x]_0^{l_2} + \frac{W^2}{24EI}\left[(x+l_1)^3\right]_0^{l_1}$$

$$= \frac{W^2 l_1^3}{24EI} + \frac{W^2 l_1^2 l_2}{8EI} + \frac{W^2}{24EI}(8l_1^3 - l_1^3)$$

$$= \frac{W^2 l_1^3}{24EI} + \frac{W^2 l_1^2 l_2}{8EI} + \frac{7W^2 l_1^3}{24EI}$$

$$= \frac{W^2 l_1^2}{24EI}[l_1 + 3l_2 + 7l_1)$$

$$= \frac{W^2 l_1^2}{24EI}(8l_1 + 3l_2)$$

Total Strain Energy $= 2 \times U_{ABCG}$

$$= 2 \times \frac{W^2 l_1^2}{24EI}(8l_1 + 3l_2)$$

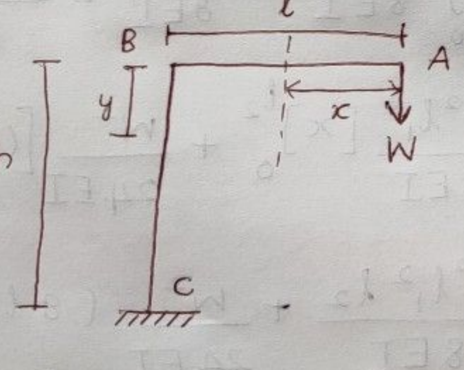
$$= \frac{W^2 l_1^2}{12EI}(8l_1 + 3l_2)$$

Vertical Deflection $G = \frac{\partial U}{\partial W}$

$$= \frac{2Wl_1^2}{12EI}(8l_1 + 3l_2)$$

$$\boxed{G = \frac{W l_1^2}{6EI}(8l_1 + 3l_2)}$$

4) In the structure shown in fig. Assuming the member to be of uniform cross-section throughout Find the strain Energy stored by the structure & Hence find the vertical deflection at end A.




Section AB

$$M_x = W \times x$$

$$U_{AB} = \int_0^l \frac{M_x^2\,dx}{2EI} = \int_0^l \frac{W^2 x^2}{2EI}\,dx = \frac{W^2 l^3}{6EI}$$

Section BC.

$$M_y = W \cdot l$$

$$U_{BC} = \int_0^h \frac{M_y^2\,dy}{2EI} = \int_0^h \frac{W^2 l^2\,dy}{2EI} = \frac{W^2 l^2 h}{2EI}$$

Total Strain Energy,

$$U = U_{AB} + U_{BC}$$

$$= \frac{W^2 l^3}{6EI} + \frac{W^2 l^2 h}{2EI}$$

$$= \frac{W^2 l^2}{6EI}(l + 3h).$$

Let $\delta$ be the deflection of end A. Then,

Work done by W = Total Strain Energy stored.

$$\frac{1}{2} W \delta = \frac{W^2 l^2}{6EI}(l + 3h)$$

$$\boxed{\delta = \frac{W l^2}{3EI}(l + 3h)}$$