Nature of quadratic form:

When the quadratic form $Q = X^T A X$ is reduced to a canonical form, it will contain only 'r' terms, if rank of A is 'r'.

Index: The no. of +ve square terms in the canonical form is called a Index of quadratic form & is denoted by p.

Signature:

The difference of no. of positive & non-positive (neg.) square terms is called the signature of quadratic form. Signature $s = 2p - r$.

Nature	Thro' Rank & Index	Thro' E.values	Thro' principle Minors
1) positive definite	$r = n$ & $p = n$ n - no. of variables	All E.values are positive	$D_1 = \|a_{11}\|$, $D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ ... $D_n = \|A\|$ All $D_1, D_2 \ldots$ are positive
2) Negative definite	$r = n$ & $p = 0$	All E.values are negative	$D_1, D_3 \ldots$ are negative $D_2, D_4 \ldots$ are positive
3) positive semi definite	$r < n$ & $p = r$	All E.values are ≥ 0 & atleast one E.value is zero	All $D_n \geq 0$ & atleast one $D_i = 0$
) Negative semi definite	$r < n$ & $p = 0$	All E.values are ≤ 0 & atleast one E.value is zero	$(-1)^n D_n \geq 0$ & atleast one $D_i = 0$
5) Indefinite	In all other cases.	positive as well as negative E.values	In all other cases.

Find the nature of quadratic form

1) $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx$.

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$D_1 = |6| = 6 > 0$

$$D_2 = \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} = 18 - 4 = 14 > 0$$

$$D_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 32 > 0$$

Hence $D_1, D_2, D_3 > 0$

$\therefore$ The gn. quadratic form is +ve definite.

2) $2x^2 + 2xy + 3y^2$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

$D_1 = |2| = 2 > 0$

$$D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5 > 0$$

$\therefore$ The gn. quadratic form is +ve definite.

3) $-x_1^2 - 2x_2^2 - 3x_3^2$

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$D_1 = |-1| = -1 < 0$

$D_2 = \begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} = 2 > 0$

$D_3 = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{vmatrix} = -6 < 0$

$D_1, D_3 < 0$, $D_2 > 0$, The gn quadratic form is -ve definite

4) $8x^2 + 7y^2 + 3z^2 - 12xy - 8yz + 4xz$.

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$D_1 = |8| = 8 > 0$

$D_2 = \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 20 > 0$

$D_3 = |A| = 0$

$\therefore D_1, D_2 > 0$ & $D_3 = 0$, the gn. quadratic form is +ve semidefinite.

1) Reduce the quadratic form $8x^2+7y^2+3z^2-12xy-8yz+4xz$ into canonical form by orthogonal transf. Find also the rank, index, sign. & nature of the quadratic form.

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

char. Eqn: $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

E. values: 0, 3, 15

E. vector: $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}; \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}; \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

N. E. vector $N = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}$

$D = N^T A N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

$Q = Y^T D Y = 3y_2^2 + 15y_3^2$

Rank : 2 (2 non-zero E. values)
Index : 2 (2 +ve E. values)
Signature : 2 ; Nature : positive semi definite

2) Reduce the quadratic form $Q = 6x^2+3y^2+3z^2-4xy-2yz+4zx$ into canonical form by an orthogonal transformation. Find its nature, rank, index & signature.

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Char. Eqn. : $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

E. values : 2, 2, 8.

E. vectors : $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$; $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$; $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

N.E. Matrix : $N = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}$

$$D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$Q = 8y_1^2 + 2y_2^2 + 2y_3^2$.

Nature : positive definite

Rank : 3

Index : 3

Signature : 3

$X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

$2l - m + n = 0$

$0l + m + n = 0$

$m = -n \Rightarrow \frac{m}{-1} = \frac{n}{1}$

$2l - m + n = 0$

$2l + n + n = 0$

$2l + 2n = 0$

$2l = -2n$

$\frac{l}{-1} = \frac{n}{1}$

$\begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$