Orthogonal transformation of a symmetric matrix to Diagonal form :

The transformation $D = N^T A N$ is known as orthogonal transformation or orthogonal reduction, where N is normalized model matrix and D is the diagonal matrix whose diagonal elts are E. values of sym. matrix A.

Methods to diagonise :

Step 1 :– Find the char. Eqn.

Step 2 :– Find the E. values & E. vectors

Step 3 :– E. vectors should be pairwise orthogonal

(ie) $X_1^T X_2 = 0$; $X_2^T X_3 = 0$; $X_3^T X_1 = 0$

Step 4 :– Normalized each E. vector X as follows.

If $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is a column vectors then normalized E. vector $X = \begin{bmatrix} x_1 / l(x) \\ x_2 / l(x) \\ x_3 / l(x) \end{bmatrix}$

where $l(x) = \sqrt{x_1^2 + x_2^2 + x_3^2}$

step 5 :- Form the normalized Model matrix N using the normalized E.vector.

step 6 :- Normalized model matrix N should be orthogonal (i.e) $NN^T = N^TN = I$

step 7 :- find $D = N^TAN$

1) Reduce the Matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ to diagonal form by orthogonal transformation

Soln: $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

step 1 :- To find the char. Eqn. $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Here $S_1 = 18$, $S_2 = 45$, $S_3 = 0$

$\therefore$ The char. eqn. is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

step 2 :- To find E.values & E.vectors.

$\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\Rightarrow \lambda = 0, 3, 15$

$\therefore$ E. values are $0, 3, 15$

Step 3: to find E. vector $(A - \lambda I)x = 0$

$$\left[\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case (i): when $\lambda = 0$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow X_1 = \begin{bmatrix} 5 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Case (ii): when $\lambda = 3$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow X_2 = \begin{bmatrix} -16 \\ -8 \\ 16 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Case (iii): when $\lambda = 15$

$$\begin{bmatrix} 7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow X_3 = \begin{bmatrix} 80 \\ -80 \\ 40 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Step 3:- to check the E. vectors are orthogonal.

$$X_1^T X_2 = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 0$$

$$X_2^T X_3 = \begin{bmatrix} 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = 0$$

$$X_3^T X_1 = \begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 0$$

Step 4:- $\therefore$ E. vectors are pairwise orthogonal.

to find the normalized E. vector

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \Rightarrow l(X_1) = \sqrt{1+4+4} = \sqrt{9} = 3$$

$\therefore$ Normalized E. vector $X_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \Rightarrow l(X_2) = \sqrt{4+1+4} = \sqrt{9} = 3$$

$\therefore$ Normalized E.vector $X_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \Rightarrow l(X_3) = \sqrt{4+4+1} = \sqrt{9} = 3$

$\therefore$ Normalized E.vector $X_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$

Step 5:-

Normalized model matrix $N = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$

Step 6:- To check N is orthogonal

(ie) $N^TN = NN^T = I$

$$N^TN = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore N$ is orthogonal.

Step 7 :- To find $D = N^T A N$

$D = N^T A N$ =

$$= \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$, the diagonal elts are E.values of A.