



DEPARTMENT OF MATHEMATICS

UNIT - I MATRIX EIGENVALUE PROBLEMS

Cayley - Hamilton Theorem :-

Every square matrix satisfies its own char. eqn

Applications :-

- (i) to calculate the +ve integral powers &
- (ii) to calculate the inverse of a square matrix A .

(i) verify that $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ satisfies its own char eqn. & find A^4 .

soln: The char. eqn. is $\lambda^2 - S_1\lambda + S_2 = 0$

$$S_1 = 0; S_2 = -5$$

\therefore The char. eqn. is $\lambda^2 - 5 = 0$.

To prove: $A^2 - 0A + 5I = 0$ or $A^2 - 5I = 0$

$$\begin{aligned} \text{Now } A^2 - 5I &= \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} - 5 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &= 0 \end{aligned}$$

\therefore The given matrix satisfies its own char. eqn.



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2) Verify Cayley-Hamilton theorem find A^4 and A^{-1}

when $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Soln: The char. Eqn. is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

Here $s_1 = 6$; $s_2 = 8$; $s_3 = 3$

\therefore the char. Eqn. is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

To prove: CHT (i) $A^3 - 6A^2 + 8A - 3I = 0$ — (1)

Now $A^3 - 6A^2 + 8A - 3I$

$$= \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}^3 - 6 \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}^2 + 8 \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

= 0

\therefore the gn. matrix satisfies its own char. Eqn.



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To find: A^4

$$\text{From (1) } A^3 = 6A^2 - 8A + 3I$$

$$A \times A^3 = A \times [6A^2 - 8A + 3I]$$

$$A^4 = 6A^3 - 8A^2 + 3A$$

$$= 6[6A^2 - 8A + 3I] - 8A^2 + 3A$$

$$= 36A^2 - 48A + 18I - 8A^2 + 3A$$

$$= 28A^2 - 45A + 18I$$

$$= 28 \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 45 \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 18 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

To find A^{-1} :

$$\text{From (1) } A^3 - 6A^2 + 8A - 3I = 0$$

$$\cdot X' A^{-1} \text{ we get, } A^2 - 6A + 8 - 3A^{-1} = 0$$

$$\Rightarrow 3A^{-1} = A^2 - 6A + 8I$$



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$$A^{-1} = \frac{1}{3} \left\{ \begin{aligned} & \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6 \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \\ & + 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \right\}$$
$$= \frac{1}{3} \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

3) use Cayley-Hamilton theorem to find the value of the matrix eqn. by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = 0$ if the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

soln: The char. Eqn. is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$S_1 = 5; S_2 = 7; S_3 = 3$$

\therefore The char. Eqn. is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$



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By CHT we get $A^3 - 5A^2 + 7A - 3I = 0$.

To find the value of $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$$\begin{array}{r}
 A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \\
 \underline{A^5 + A} \\
 A^3 - 5A^2 + 7A - 3I \\
 \underline{A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I} \\
 A^8 - 5A^7 + 7A^6 - 3A^5 \\
 \underline{A^4 - 5A^3 + 8A^2 - 2A} \\
 A^4 - 5A^4 + 7A^2 - 3A \\
 \underline{A^2 + A + I}
 \end{array}$$

$$\therefore A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= (A^3 - 5A^2 + 7A - 3I)(A^5 + A) + (A^2 + A + I)$$

$$= 0(A^5 + A) + (A^2 + A + I) \quad [\because \text{CHT}]$$

$$= A^2 + A + I$$

$$\begin{aligned}
 \text{Now } A^2 + A + I &= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}
 \end{aligned}$$