(14 marks).

REDUCTION OF QUADRATIC EQUATIONS TO CANONICAL FORM :- (2 marks).

Definition :- A homogeneous polynomial of degree 2 with any number of variables are known as quadratic form.

$$X^2 + Y^2 + Z^2 + 2XY + 2YZ + 2ZX = 0.$$

Formula :- i.e. quadratic form to matrix form.

$$\begin{bmatrix} \text{Co-efficient of } x^2 & \frac{1}{2}\text{ co-efficient of } xy & \frac{1}{2}\text{ co-efficient of } zx \\ \frac{1}{2}\text{ co-efficient of } yz & \text{Co-efficient of } y^2 & \frac{1}{2}\text{ co-efficient of } yz \\ \frac{1}{2}\text{ co-efficient of } zx & \frac{1}{2}\text{ co-efficient of } zy & \text{Coefficient of } z^2 \end{bmatrix}$$

CANONICAL FORM :- If a quadratic form $Q = X^TAX$ can be reduced by a non-singular linear transformations $X = Ny$ to $Q = y^TAy$ Y^TDY, where Y is column vector then the form

$[\cancel{Y = D}]$ $Q = Y^T D Y$ is known as canonical form.

INDEX (P) :-

Index = no. of positive square terms in the canonical form.

RANK (r) :-

Rank = no. of non-zero eigenvalues in the canonical form.

SIGNATURE (s) :-

Signature = difference between number of positive and negative square terms in the canonical form.

i.e. $s = 2p - r$.

(2mrks).

NATURE OF A QUADRATIC FORM :-

1) Positive definite :-

All the eigenvalues of A are positive.

Example :- $\lambda = 1, 2, 3$.

2) Negative definite :-

All the eigenvalues of A are negative.

Example :- $\lambda = -1, -2, -2$.

3) Positive semi-definite :-

All the eigenvalues of A are non-negative and at least one eigenvalue is zero.

Example :- $\lambda = 0, 1, 2$.

4) Negative semi-definite :-

All the eigenvalues of A are non-positive and at least one eigenvalue is zero.

Example :- $\lambda = 0, -1, -2$.

5) Indefinite :-

Some eigenvalues are positive and some eigenvalues are negative

Example :- $\lambda = -1, 1, 2$

$\lambda = -2, -1, 2$.

23-09-19

REDUCED THE QUADRATIC FORM TO CANONICAL FORM.

Example 1 : Write the matrix of the quadratic form to $2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2$
(2m)

Soln:-

The matrix form of the quadratic equation is

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{bmatrix}.$$

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Example 2: Write the quadratic form corresponding to the following symmetric matrix $\begin{bmatrix} 0 & -1 & 2 \\ -1 & 1 & 4 \\ 2 & 4 & 3 \end{bmatrix}$.

Ans:- The quadratic form of the symmetric matrix is

$0x_1^2 + x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_1x_3 + 8x_2x_3$.

Example 3: Reduced the quadratic form ~~$[x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3 + 2x_1$ to canonical form$]$~~

$8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$.

into canonical form by orthogonal transformation and discuss its nature.

Ans:- The matrix form of the quadratic equation is.

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}.$$

Step 1:- Characteristic equation, Eigenvalue and Eigenvector.

The characteristic equation is $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$.

D_1 = sum of the leading diagonal elements.

$= 8 + 7 + 3$.

$= 18$.

D_2 = sum of the minors of the leading diagonal elements.

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}.$$

$= (21-16) + \cancel{(1820)}\ (24-4) + (56-36)$.

$= 5 + 20 + 20$.

$= 45$.

$D_3 = \det. A$.

$$= \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$= 8(21-16) + 6(-18+8) + 2(24-14)$.

$= 8 \times 5 + 6 \times (-10) + 2(10)$.

$= 40 - 60 + 20$.

$= 0$.

$\therefore$ The characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$.

Eigenvalue :- $\mathcal{I}$ $\lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$.

$\therefore$ The value of λ is $0, 3, 15$.

Eigenvectors :- $[A - \lambda I]X = 0$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

When $\lambda = 0$,

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$8x_1 - 6x_2 + 2x_3 = 0$.

$-6x_1 + 7x_2 - 4x_3 = 0$.

$2x_1 - 4x_2 + 3x_3 = 0$

$$\begin{array}{cccc} & \overline{x_2} & & \\ -6 & 2 & 8 & -6 \\ 7 & -4 & -6 & 7 \\ \multicolumn{2}{c}{\underline{}\,x_1} & \multicolumn{2}{c}{\underline{}\,x_3} \end{array}$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & 8 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}.$$

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}.$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}.$$

$$\therefore X_1 = \begin{bmatrix} 10 \\ 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

When $\lambda = 3$.

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$5x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 + 4x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 + 0x_3 = 0.$$

$$\begin{array}{cccc} -6 & 2 & 5 & -6 \\ 4 & -4 & -6 & 4. \end{array}$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & 5 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}.$$

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}.$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}.$$

$$\therefore X_2 = \begin{bmatrix} 16 \\ 8 \\ -16 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

When $\lambda = 15$.

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$-7x_1 - 6x_2 + 2x_3 = 0.$$

$$-6x_1 - 8x_2 - 4x_3 = 0.$$

$$2x_1 - 4x_2 - 12x_3 = 0.$$

$$\begin{array}{cccc} -6 & \overbrace{2 \quad\quad -7}^{x_2} & & -6. \\ \underbrace{-8 \quad\quad -4}_{x_1} & & \underbrace{-6 \quad\quad -8}_{x_3}. & \end{array}$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & -7 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}.$$

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}.$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}.$$

$$\therefore X_3 = \begin{bmatrix} 40 \\ -40 \\ 20 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Characteristic eq.^{n}.	Eigenvalue	Eigenvector.
$\lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$.	$\lambda = 0, 3, 15$.	$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. $X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

Step 2 : Check orthogonal condition.

$$X_1^T X_2 = 0 \Rightarrow \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 2 + 2 - 4 = 0.$$

$$X_2^T X_3 = 0 \Rightarrow \begin{bmatrix} 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = 4 - 2 - 2 = 0.$$

$$X_3^T X_1 = 0 \Rightarrow \begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 2 - 4 + 2 = 0.$$

Step 3 :- Find normalized matrix.

Eigenvector	$l(x) = \sqrt{x_1^2 + x_2^2 + x_3^2}$	Normalized vector
$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$	$l(x_1) = \sqrt{1^2 + 2^2 + 2^2}$. $= 3$.	$N_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$
$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$	$l(x_2) = \sqrt{2^2 + 1^2 + (-2)^2}$ $= 3$.	$N_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$.
$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.	$l(x_3) = \sqrt{2^2 + (-2)^2 + 1^2}$. $= 3$.	$N_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$.

Step 4 :- Model matrix.
Medal.

$$N = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}.$$

$$= \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}.$$

$$N^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}.$$

Step 5 :- $N^T A N = D$

$$\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}.$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}.$$

Step 6 :- Canonical form.

i.e., $Y^T D Y$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

∴ The canonical form is $0y_1^2 + 3y_2^2 + 15y_3^2$.

∴ The nature of the quadratic form is positive semi definite.

Rank (r) = no. of non-zero element $\Rightarrow r = 2$.

Index (p) (~~Signature (s)~~) = no. of ~~non~~ square terms in canonical form.

$\Rightarrow s = 3.$

∴ Signature $(s) = 2p - r$

$= 6(p) - 2$

$= 4.$

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