Date :- 17-09-19
Day :- Tuesday.

UNIT-II.

ORTHOGONAL TRANSFORMATION OF SYMMETRIC MATRIX OF DIAGONAL FORM.

The transformation $N^T A N = D$ is known as orthogonal transformation or reduction.

Where N is normalized matrix.

D is diagonal matrix whose diagonal elements are eigenvalues of the given matrix.

Methods through diagonalized a symmetric matrix by orthogonal transformation :-

Step 1 :- Find characteristic equation and eigenvalues and so eigenvectors of a matrix A.

Step 2 :- Check orthogonal condition

i.e. $X_1^T X_2 = 0; \; X_2^T X_3 = 0 \; ; \; X_3^T X_1 = 0.$

Step 3 :- To find normalized eigenvector if $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then the normalized vector N =

$$N = \begin{bmatrix} \dfrac{x_1}{l(x_1)} \\ \dfrac{x_2}{l(x_2)} \\ \dfrac{x_3}{l(x_3)}. \end{bmatrix}, \text{ where } l(x) = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Step 4 :- Find model matrix i.e. ~~$M = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$~~ $N = [N_1 \, N_2 \, N_3]$

Step 5 :- Find $N^T A N = D$, where D = diagonal matrix.

Example 1 :- ~~[Diagoler]~~ Diagonalized the matrix by autogonal transformation for

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}.$$

Step 1: Characteristic eq.^{n}, Eigenvalue & Eigenvectors.

The characteristic equation is $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

$D_1 = 8 + 7 + 3 = 18.$

$$D_2 = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$= (21-16) + (24-4) + (56-36).$

$= 5 + 20 + 20.$

$= 45.$

$D_3 = |A|.$

$$= \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}.$$

$= 8(21-16) + 6(-18+8) + 2(24-14).$

$= 8 \times 5 + 6 \times (-10) + 2(10).$

$= 40 - 60 + 20$

$= 0.$

$\therefore$ The characteristic eq.^{n} is $\lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$

Eigenvalue :- $\lambda^3 - 18\lambda + 45\lambda - 0 = 0$

The value of λ is $0, 3, 15$.

Eigenvector :-

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

When $\lambda = 0$,

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$8x_1 - 6x_2 + 2x_3 = 0.$$
$$-6x_1 + 7x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 + 3x_3 = 0.$$

$$\begin{matrix} -6 & 2 & 8 & -6 \\ 7 & -4 & -6 & 7. \end{matrix}$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & 8 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}.$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}.$$

$$\therefore X_1 = \begin{bmatrix} 10 \\ 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

When $\lambda = 3$.

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$5x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 + 4x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 + 0x_3 = 0.$$

$$\begin{matrix} -6 & 2 & 5 & -6 \\ 4 & -4 & -6 & 4 \end{matrix}.$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & 5 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}.$$

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}.$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\therefore X_2 = \begin{bmatrix} 16 \\ 8 \\ -16 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

When $\lambda = 15$.

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$-7x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 - 8x_2 - 4x_3 = 0$.

$2x_1 - 4x_2 - 12x_3 = 0$.

$$\begin{matrix} -6 & 2 & -7 & -6 \\ -8 & -4 & -6 & -8 \end{matrix}$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & -7 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}.$$

$$\begin{array}{r} 56 \\ 36 \\ \hline 92 \end{array}.$$

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{\cancel{-}56+36}.$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{\cancel{-9}20}.$$

$$\therefore X_3 = \cancel{\begin{bmatrix} 40 \\ 40 \\ -92 \end{bmatrix}}. \begin{bmatrix} 40 \\ -40 \\ 20 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Step 2 : Check orthogonal condition

$$X_1^T X_2 = 0 \Rightarrow (1 \quad 2 \quad 2)\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = -2-2+4 = 0.$$

$4-2-2$

$$X_2^T X_3 = 0 \Rightarrow (-2 \quad -1 \quad -2)\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \cancel{-4+2+2=0}.$$

$$X_3^T X_1 = 0 \Rightarrow (2 \quad -2 \quad 1)\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 2-4+2 = 0.$$

Step 3 : Find normalised matrix.

Eigenvector	$f(x) = \sqrt{x_1^2 + x_2^2 + x_3^2}$	Normalized vector.
$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$	$l(x_1) = \sqrt{1^2 + 2^2 + 2^2} = 3.$	$N_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}.$
$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$	$l(x_2) = \sqrt{2^2 + 1^2 + (-2)^2} = 3.$	$N_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}.$
$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$	$l(x_3) = \sqrt{2^2 + (-2)^2 + 1^2} = 3$	$N_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$

Step 4 : Model matrix.

$$N = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}.$$

(1) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} ; \lambda = -1, 1, 4$

$$N = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}.$$

(2) $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \lambda = 0, 1, 3$

$$N^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}:$$

(3) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \lambda = -2, 3, 4$

(4) $\begin{pmatrix} 16 & -2 & 5 \\ -2 & 2 & 3 \end{pmatrix}$

Step 5 :- $N^T A N = D$.

$$\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}.$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}.$$

18-09-19.