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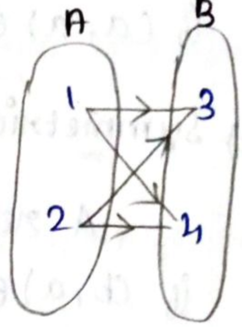
Relations :

If A and B be two non empty sets, then the relation R from A to B is a subset such that

$R \subseteq A \times B$, $(a,b) \in R$ (or) $a R b$



Eg:- $A = \{1,2\}$ $B = \{3,4\}$

$A \times B = \{(1,3),(1,4)(2,3),(2,4)\}$

Example: 1

Let R be a relation given by $R = \{(a,b), a = b-2, b>6\}$

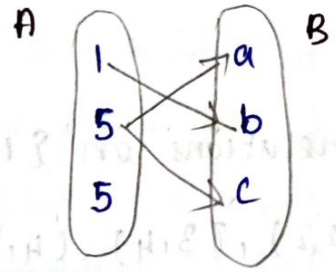
Choose the correct options :

i) $(2,4) \in R$ → This is not a relation

ii) $(3,8) \in R$ → This not a relation

iii) $(6,8) \in R$ → This is a relation

Eg:- $\{(1,b),(5,a),(5,c)\}$



Example: 2

Consider the relations on the set of Integer

$R_1 = \{(a,b) / a \leq b\}$ (1,1) (1,2)

$R_2 = \{(a,b) / a > b\}$ (2,1)

$R_3 = \{(a,b) / a = b$ (or) $a = -b\}$ (1,1) (1,-1) (2,-2)

$R_4 = \{(a,b) / a = b\}$ (1,1)

$R_5 = \{(a,b) / a = b+1\}$ (2,1)

$R_6 = \{(a,b) / a+b \leq 3\}$

Which of these contain each of the pairs (1,1), (1,2), (2,1) (1,-1) (2,-2) ?

Properties of Relations :

types of relations

1) **Reflexive** :

A relation R on a set A is called reflexive if $(a,a) \in R$ for every element $a \in A$

2) **Symmetric and Anti - symmetric** :

A relation R on a set A is called symmetric if $(b,a) \in R$, whenever $(a,b) \in R \ \forall \ a,b \in A$

A relation R on a set A such that $\forall \ a,b \in A$, if $(a,b) \in R$ and $(b,a) \in R$ then $a=b$ is called antisymmetric.

3) **Transitive** :

A relation R on a set A is called transitive whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ $\forall \ a,b,c \in A$ (a,b) (b,c) (a,c)

Example: 1

Consider the following relations on {1,2,3,4}

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1)(4,4)\}$

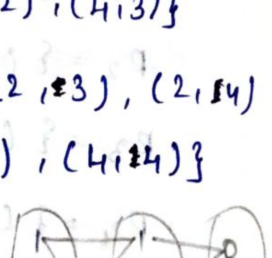
$R_2 = \{(1,1)(1,2),(2,1)\}$

$R_3 = \{(1,1),(1,2),(1,4),(2,1),(3,3),(4,1),(4,4)\}$

$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$

$R_5 = \{(1,1),(1,2),(1,3),(1,4)(2,2)(2,3),(2,4)$
$(3,3),(4,4)\}$

$R_6 = \{(3,4)\}$



i) which of these solutions are reflexive ?

The relations R_3 and R_5 are reflexive because they both have contain $(a, b) \to (1,1), (2,2), (3,3)$ $(4,4)$

ii) which of the relation are symmetric and Anti symmetric ?

The relation R_2 and R_3 are symmetric because, (b, a) R $(a, b) = (2,1)$ R $(1,2)$

The relations R_1, R_2, R_3 are antisymmetric because $a \leq b$ and $b \leq a \Rightarrow a = b$

iii) which of the relations are transitive ?

R_4, R_5 and R_6 are transitive because,

$((a, b) \in R \cap (b, c) \in R) \rightarrow (a, c) \in R)$

$(3,2) \in R \wedge (2,1) \in R \rightarrow (3,1) \in R$

Definition :.

Let R be a relation from a set A to set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$ for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$ it is denoted $S \circ R$

Example 1:-

What is the composite of the relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ and S in the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1,0), (2,0)$ $(3,0)$ $(3,2)$ $(4,1)\}$? $(R = \{(1,1)$ $(1,4), (2,3), (3,1), (3,4)\}$

Sol:

$\cancel{S \circ R} = \{(1,0), (1,1), (2,0), (2,2), (3,0), (3,1)\}$

$R \circ S$

Matrix representation :.

A relation between finite sets can be represented using a zero - one matrix. suppose that R is a solution from $A = \{a_1, a_2, \cdots a_n\}$ and $B = \{b_1, b_2, \cdots b_n\}$ the relation R can be represented by the matrix $M_R = [m_{ij}]$

where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \in R \end{cases}$$

Example:-1

Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$ let R be the relations from A to B containing (a, b) if $a \in A$, $b \in B$ and $a > b$. what is the matrix representing R if $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and $b_1 = 1$ $b_2 = 2$?

Sol:

$R = \{(2,1), (3,1), (3,2)\}$

$M_R = \underset{\text{Row}}{3} \times \overset{\text{column}}{2}$ matrix

$$M_R = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array}$$

Note:-

i) Reflexive if $m_{ij} = 1$

ii) Symmetric if $m_{ij} = 1$

iii) Anti symmetric if [$m_{ij} = 0$ then $m_{ji} = 1$] (or)

[$m_{ij} = 1$, then $m_{ji} = 0$]

where $i \neq j$

Example: 2

Suppose that the relation R on a set is represented by the matrix $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is R is reflexive symmetric and Anti-Symmetric?

Sol:

R is relexive, because m_{11}, m_{22}, m_{33} are 1

R is symmetric, because $m_{12} = m_{21}$

R is not a Anti-symmetric.

Example: 3

Suppose that the relation R_1 and R_2 on a set A are represented by the matrices.

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Sol:

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Composite of relations using matrices:

Example: 1

Let $A = \{2, 3, 5, 8\}$, $B = \{4, 6, 16\}$, $C = \{1, 4, 5, 7\}$

Let $R = \{(a, b) = a/b\}$, and $S = \{(b, c) : b \le c\}$ be relations from A to B and B to C. Find the composite relations $S \circ R$?

sol:

$A = \{2,3,5,8\}$ $R = \{(a,b) : a/b\}$

$B = \{4,6,16\}$ $S = \{(b,c) : b \leq c\}$

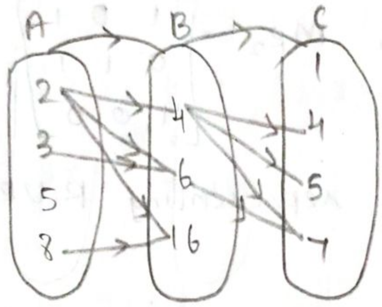
$C = \{1,4,5,7\}$

$R = \{ (2/4)(3/4)(5/4)(8/4)$
$(2/6)(3/6)(5/6)(8/6)$
$(2/16)(3/16)(5/16)(8/16)\}$

$R = \{(2,4)(2,6)(2,16)(3,6)(8,16)\}$

$S = \{(4 \leq 1)(6 \leq 1)(16 \leq 1)$
$(4 \leq 4)(6 \leq 4)(16 \leq 4)$
$(4 \leq 5)(6 \leq 5)(16 \leq 5)$
$(4 \leq 7)(6 \leq 7)(16 \leq 7)\}$

$S = \{(4,4)(4,5)(4,7)(6,7)\}$



$S_oR = \{(2,4)(2,5)(2,7)(3,7)\}$

$$M_{ROS} = \begin{array}{c} \\ 2 \\ 3 \\ 5 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 4 & 5 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

18/12 Example:-

let R be the relation represented by the matrix $M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ Find the matrix representing

a) R^{-1} b) $\bar{R}$ c) R^4

Sol:

$$M_R = \begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 1 & 1 \\ b & 1 & 1 & 0 \\ c & 1 & 0 & 1 \end{array}$$

$R = \{(a,b), (a,c), (b,a), (b,b), (c,a), (c,c)\}$

$$R^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \bar{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$R^4 = R^2 \cdot R^2$

$$R^2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$R^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$R^4 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$R^4 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Transitive closure:-

Let x be any finite set and R be a relation in x the relation $R^+ = R \cup R^2 \cup R^3 \cup \ldots$ in x is called the Transitive closer of R in x.

Example:-

Let $A = \{1, 2, 3, 4\}$ and $R = \{(1,2)(2,3)(3,3),(3,4)(4,2)\}$ be a relation defined on A. Find the Transitive closer of R.

Sol:

$A = \{1,2,3,4\}, R = \{(1,2),(2,3),(3,3),(3,4),(4,2)\}$

$$M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$M_{R^2} = M_R \circ M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$M_R^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$M_{R^3} = M_{R^2} \cdot M_R$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$M_R^4 = M_R^3 \cdot M_R$$

$$= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$R^4 = R \cup R^2 \cup R^3 \cup R^4$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$R^4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$R^4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$R^4 = \{(1,2)\ (1,3)\ (1,4)\ (2,2)\ (2,3)\ (2,4)\ (3,2)\ (3,3)\ (3,4)\ (4,2)\ (4,3)\ (4,4)\}.$$