Unit - 1

Matrices

Definition

=> A system of m,n numbers arranged in an rectangular array along m rows and n columns and is called m×n matrix

=> That is

$$A = \begin{bmatrix} a_{11}, a_{12} \cdots a_{1n} \\ a_{21}, a_{22} \cdots a_{2n} \\ \vdots \\ a_{m1}, a_{m2} \cdots a_{mn} \end{bmatrix}$$

=> Here A is a matrix of order m×n matrix where m means no of rows N means no of columns and m, N means no of element of the matrix.

=> matrix A is denoted by $A = [a_{ij}]$

Characterstic equation :-

=> Let A be a given matrix. Let λ be a scalar matrix then the equation $|A - \lambda I| = 0$ is called characterstic equation of a matrix A.

Easy method to find characterstic equation :.

1) The characterstic equation of A (3×3) matrix is $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

2) where D_1 = sum of the leading diagonal element

   D_2 = sum of the minus of the leading diagonal elementes

   D_3 = determinate A

3) For (2×2) matrix the characterstic equation is $\lambda^2 - D_1\lambda + D_2 = 0$

Where D_1 = Sum of the leading diagonal element

D_2 = Determinate A.

Problem : 1

Find the characteristic equation of $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

Sol:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

The characterstic Equation is $\lambda^2 - D_1\lambda + D_2 = 0$

D_1 = sum of main diagonal element

$= 1+2 \Rightarrow 3$

$D_2 = |A|$

$= \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = (2-0) \Rightarrow 2$

∴ The characterstic Equation is $\lambda^2 - 3\lambda + 2 = 0$

Problem : 2

Find the characterstic equation of $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Sol:

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

The characterstic Equation is $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

D_1 = sum of main diagonal element

$= -2+1+0 \Rightarrow \boxed{-1}$

D_2 = sum of minus of diagonal element

$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$

$= (0-12)+(0-3)+(-2-4)$

$= -12-3-6$

$D_2 = \boxed{-21}$

$$D_3 = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} - 2\begin{vmatrix} 2 & -6 \\ -1 & 0 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix}$$

$= -2(-12) - 2(-6) - 3(-4+1)$

$= 24 + 12 + 9$

$\boxed{= 45}$

$\therefore$ The characterstic Equation is $\lambda^3 + \lambda^2 21\lambda - 45 = 0$

problem :3

If the characterstic equation of $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 1 \\ -7 & 2 & 3 \end{bmatrix}$ is $\lambda^3 + a\lambda^2 + b\lambda - 12 = 0$ then find the value of a and b

Sol:

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 1 \\ -7 & 2 & 3 \end{bmatrix} \qquad \lambda^3 + a\lambda^2 + b\lambda - 12 = 0$$

we know that

a = sum of main diagonal element

$= 2+2+3 = 7$

b = sum of minius of diagonal element

$$= \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -7 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix}$$

$= (6-2) + (6+7) + (4-4)$

$= 18$

$\therefore$ The characteristic equation is

$$\lambda^3 + 7\lambda^2 + 10\lambda - 12 = 0$$

Then the value of $a = 7$

$b = 12$

Problem : 4

If the characteristic equation of $\begin{bmatrix} 5 & 2 & -3 \\ 0 & 0 & 8 \\ 0 & 0 & 7 \end{bmatrix}$ is

$\lambda^3 - 12\lambda^2 + 35\lambda - k = 0$

Sol:

$$A = \begin{bmatrix} 5 & 2 & -3 \\ 0 & 0 & 8 \\ 0 & 0 & 7 \end{bmatrix}$$

we know that

k = Determinant of A

$$A = \begin{bmatrix} \overset{+}{5} & \overset{-}{2} & \overset{+}{-3} \\ 0 & 0 & 8 \\ 0 & 0 & 7 \end{bmatrix}$$

$= 5(0-0) - 2(0-0) - 3(0-0)$

$= 0$

The value $k = 0$

The characteristic equation

$\lambda^3 - 12\lambda^2 + 35\lambda - 0 = 0$

Problem : 5

Write two matrices with $\lambda^2 - 7\lambda + 6 = 0$ as the characteristic equation.

Sol:

$\lambda^2 - 7\lambda + 6 = 0$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$ are the two matrices.

Rank of a matrix

$\Rightarrow$ The matrix A is said to be of rank r, if

i) A as atleast one minor of order r which does not vanish.

ii) Every minor of A of order $(r+1)$ and higher order vanishes

In other words, the rank of a matrix is the order of any highest order non-vanishing minor of the matrix.

It is denoted by $\rho(A)$

Problem 1:

Type: 1

Find the rank of a matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$

Sol:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{vmatrix} = 1(21-20) - 2(14-12) + 3(10-9)$$

$$= 1(1) - 2(2) + 3(1)$$

$$= 1 - 4 + 3 \Rightarrow 0$$

$|A| = 0$

$$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$$

$\boxed{\therefore \rho(A) = 2}$

Problem: 2

Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 6 & 3 \end{bmatrix}$

Sol:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 6 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 6 - 6 = 0$$

$$\begin{vmatrix} 2 & 4 \\ 6 & 3 \end{vmatrix} = 6 - 24 = -18 \neq 0$$

$\boxed{\therefore \rho(A) = 2}$

Type : 2

Find the rank of matrix $A = \begin{pmatrix} 1 & 3 & 5 & -1 \\ 2 & 1 & -2 & 8 \\ 0 & 5 & 12 & -10 \end{pmatrix}$

sol:

$$A = \left|\begin{array}{ccc|c} 1 & 3 & 5 & -1 \\ 2 & 1 & -2 & 8 \\ 0 & 5 & 12 & -10 \end{array}\right|_{3\times 4}$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & -1 \\ 0 & -5 & -12 & 10 \\ 0 & 5 & 12 & -10 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & -1 \\ 0 & -5 & -12 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$\boxed{\therefore \rho(A) = 2}$

Problem : 3

Find the rank of matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 3 & 1 & 2 \end{bmatrix}$

sol:

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 1 & 3 & 2 \end{bmatrix} \quad C_1 \leftrightarrow C_2$$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$\boxed{\therefore \rho(A) = 1}$

Problem: 4

Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

Sol:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 3 & 4 & 5 \end{bmatrix} \quad R_2 = R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix} \quad R_3 = R_3 - 3R_1 \qquad \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\cancel{\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix} \quad R_3 \to R_3 + R_1}$$

$$\boxed{\therefore \rho(A) = 2}$$

problem: 5

Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$

Sol:

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{vmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -2 \end{bmatrix} \quad \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 + 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\boxed{\therefore \rho(A) = 3}$$

Inverse of matrix:

problem: 1

Let A be the inverse matrix and the determinant of A be $|A|$ then the inverse is defined as i.e $A^{-1} = \frac{adj A}{|A|}$

Note : i) $Adj A = [\text{Cofactor of } A]^T$

ii) To find cofactor of $A = \begin{bmatrix} A_{11} & -A_{12} & A_{13} \\ -A_{21} & A_{22} & -A_{23} \\ A_{31} & -A_{32} & A_{33} \end{bmatrix}$

iii) where A_{11} = minor of A_{11}

problem: 2

Find the inverse of A when $A = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

Sol:

$$|A| = \begin{vmatrix} 2 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix}$$

$= 2(1) + 2(-2) + 1(-1)$

$= 2 - 4 - 1$

$= -3 \neq 0$

$\therefore A^{-1}$ exist.

co-factor of $A = \begin{bmatrix} A_{11} & -A_{12} & A_{13} \\ -A_{21} & A_{22} & -A_{23} \\ A_{31} & -A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

The cofactor of $A_{11} = (1-0) \Rightarrow 1$

The cofactor of $A_{12} = (0-2) \Rightarrow -2$

The cofactor of $A_{13} = (0-1) \Rightarrow -1$

The cofactor of $A_{21} = (-2-0) \Rightarrow -2$

The cofactor of $A_{22} = (2-1) \Rightarrow -2$

The cofactor of $A_{23} = (0+2) \Rightarrow 2$

The cofactor of $A_{31} = (-4-1) \Rightarrow -5$

The cofactor of $A_{32} = (4-0) \Rightarrow 4$

The cofactor of $A_{33} = (2-0) \Rightarrow 2$

$$\text{Co-factor} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -5 & 4 & 2 \end{bmatrix}^T$$

$$\text{adj } A = [\text{co-factor}]^T$$

$$= \begin{bmatrix} 1 & 2 & -5 \\ 2 & 1 & -4 \\ -1 & -2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-3} \begin{bmatrix} 1 & 2 & -5 \\ 2 & 1 & -4 \\ -1 & -2 & 2 \end{bmatrix}$$

Problem : 3

Find A^{-1} if $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Sol: $|A| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$

$= 1(3-0) - 2(-1-0) - 2(2-0)$

$= 3+2-4 \Rightarrow 1 \neq 0$

$\therefore A^{-1}$ exist.

$$\text{co-factor of } A = \begin{bmatrix} A_{11} & -A_{12} & A_{13} \\ -A_{21} & A_{22} & -A_{23} \\ A_{31} & -A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

The cofactor of A_{11} = $(3-0) \Rightarrow 3$

The cofactor of A_{12} = $(-1-0) \Rightarrow -1$

The cofactor of A_{13} = $(2-0) \Rightarrow 2$

The cofactor of A_{21} = $(2-4) \Rightarrow -2$

The cofactor of A_{22} = $(1-0) \Rightarrow 1$

The cofactor of A_{23} = $(-2-0) \Rightarrow -2$

The cofactor of A_{31} = $(0+6) \Rightarrow 6$

The cofactor of A_{32} = $(0-2) \Rightarrow -2$

The cofactor of A_{33} = $(3+2) \Rightarrow 5$

$$\text{cofactor} = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}$$

$$\text{Adj } A = (\text{cofactor})^T$$

$$= \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

problem : 4

Find A^{-1} of $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$

sol: $|A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix}$

$= 1(-28+30) - 0(-21-0) - 1(-18-0)$

$= 2+18$

$|A| = 20 \neq 0$

$\therefore A^{-1}$ is exist

$$\text{co factor of } A = \begin{bmatrix} A_{11} & -A_{12} & A_{13} \\ -A_{21} & A_{22} & -A_{23} \\ A_{31} & -A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$$

The cofactor of A_{11} is $= (-28+30) \Rightarrow 2$

The cofactor of A_{12} is $= (-21-0) \Rightarrow -21$

The cofactor of A_{13} is $= (-18-0) \Rightarrow -18$

The cofactor of A_{21} is $= (0-6) \Rightarrow -6$

The cofactor of A_{22} is $= (-7-0) \Rightarrow -7$

The cofactor of A_{23} is $= (-6-0) \Rightarrow -6$

The cofactor of A_{31} is $= (0+4) \Rightarrow 4$

The cofactor of A_{32} is $= (5+3) \Rightarrow 8$

The cofactor of A_{33} is $= (4-0) \Rightarrow 4$

$$\text{cofactor} = \begin{bmatrix} 2 & 21 & -18 \\ 6 & -7 & 6 \\ 4 & -8 & 4 \end{bmatrix}$$

$$\text{adj } A = (\text{co-factor})^T = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{20} \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$