Example 2:

Write the quadratic form corresponding to the following symmetric matrix $\begin{bmatrix} 0 & -1 & 2 \\ -1 & 1 & 4 \\ 2 & 4 & 3 \end{bmatrix}$.

Sol

The quadratic form of symmetric matrix is

$$0x_1^2 + x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_1x_3 + 2x_2x_3$$

Example 3:

Reduce the quadratic form
$8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$ into canonical form by orthogonal transformation and discuss its nature

The matrix form of the quadratic equation is

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Find characteristic equation, eigenvalue & eigenvector

$$\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0$$

$c_1 =$ Sum of main diagonals

$= 8+7+3 = 18$

$c_2 =$ Sum of minor of main diagonals

$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$

$= (21-16) + (24-4) + (56-36)$

$= 5 + 20 + 20 = 45$

$$c_3 = |A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + 6\begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix} + 2\begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}$$

$$= 8(21-16) + 6(-18+8) + 2(24-14)$$

$$= 8(5) + 6(-10) + 2(10)$$

$$= 40 - 60 + 20 = 0$$

$$\lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\lambda = 0, \quad \lambda = 3, \lambda = 5$$

characteristic equation

The eigen vector x corresponding to the eigen value λ

$$[A - \lambda I]\, x = 0$$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

case i) $\lambda = 0$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{matrix} 8 & -6 & 2 \\ -6 & 7 & -4 \end{matrix}$$

$$\begin{matrix} & & x_1 & \\ 8 & -6 & 2 & 8 \\ -6 & 7 & -4 & -6 \\ & x_3 & & x_2 \end{matrix}$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & 8 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Case ii) $\lambda = 3$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{cccc} & & x_1 & \\ 5 & -6 & 2 & 5 \\ -6 & 4 & -4 & -6 \\ \cancel{2} & \cancel{-4} & \cancel{0} & \\ & x_3 & & x_2 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & 5 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Case iii) $\lambda = 15$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{array}{cccc} & & x_1 & \\ -7 & -6 & 2 & -7 \\ -6 & -8 & -4 & -6 \\ & x_3 & & x_2 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & -7 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

$$\frac{x_1}{+24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Step 2: Check orthogonal condition

$$X_1^T X_2 = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$= 2+2-4 = 0$$

$$X_2^T X_3 = \begin{bmatrix} 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$= 4-2-2 = 0$$

$$X_3^T X_1 = \begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$= 2-4+2 = 0$$

Step 3: Find the normalized ~~factor~~ Matrix

Eigenvectors	$l(x) = \sqrt{x_1^2 + x_2^2 + x_3^2}$	Normalized matrix
$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$	$l(x_1) = \sqrt{9} = 3$	$N_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$
$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$	$l(x_2) = \sqrt{9} = 3$	$N_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$
$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$	$l(x_3) = \sqrt{9} = 3$	$N_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$

Step 4: Model Matrix $N = [N_1 \; N_2 \; N_3]$

$$N = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

Step 5:

$$N^T A N = D$$

$$N^T A N = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

$$N^T A N = D(0, 3, 15)$$

where the diagonal elements are 0, 3, 15

Step 6:

The canonical form is

$Y^T D Y$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\underset{1\times 3}{\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}$$

∴ The canonical form is

$$0y_1^2 + 3y_2^2 + 15y_3^2$$

The nature of quadratic form is

Positive semi definite

Rank (r) = no. of non-zero elements

= 2

Index p = no of +ve square terms in canonical form

= 2

Signature = Diff b/w +ve square terms and -ve square terms.

= 2 − 0 = 2

Example: 4

Reduce the quadratic to canonical form of $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ and discuss about its nature.

Sol

The matrix form of the quadratic eqn is

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Step 1:

Find C.E, EV & EVc

The characteristic equation is

$\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

D_1 = Sum of the leading diagonal elements

= 1+2+1

D_2 = Sum of the minor of the leading "

$= \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$

$= (2-1) + (1) + (2-1)$

= 3

$D_3 = |A|$

$$= \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1(2-1) + 1(-1+0)$$

$$= 1 - 1 = 0$$

C.E is $\lambda^3 - 4\lambda^2 + 3\lambda - 0 = 0$

$$\lambda(\lambda^2 - 4\lambda + 3) = 0$$

$$\lambda = 0, \quad \lambda = 1, \quad \lambda = 3$$

3

$-1 \quad -3$

The characteristic equation is

$$\lambda^3 - 4\lambda^2 + 3\lambda - 0 = 0$$

Eigen vector

$$\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case i) $\lambda = 0$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & -1 \end{matrix}$$

$$\frac{x_1}{-1+0} = \frac{x_2}{0-1} = \frac{x_3}{2-1} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

When $\lambda = 3$

$$\begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} -2 & -1 & 0 & -2 \\ -1 & -1 & 1 & -1 \end{matrix}$$

$$\frac{x_1}{-1-0} = \frac{x_2}{0+2} = \frac{x_3}{2-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$X_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

When $\lambda = 1$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} 0 & -1 & 0 & 0 \\ -1 & 1 & 1 & -1 \end{matrix}$$

$$\frac{x_1}{-1-0} = \frac{x_2}{0-0} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$X_3 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

Step 2 Check orthogonal condition

$$x_1^T x_2 = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = 1 + 0 - 1 = 0$$

$$x_2^T x_3 = \begin{bmatrix} -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = 1 + 0 - 1 = 0$$

$$x_3^T x_1 = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = 1 - 2 + 1 = 0$$

Step 3 : Find normalized vector

Eigenvector $\quad l(x) = \sqrt{x_1^2 + x_2^2 + x_3^2} \quad$ Normalized vector

$$x_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad l(x_1) = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \quad N_1 = \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$x_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \quad l(x_2) = \sqrt{(-1)^2 + 0 + (-1)^2} = \sqrt{2}$$

$$x_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Step 4 Model Matrix

$$N = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

Step 5: $N^T A N = D$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Step 6:

Canonical form

$Y^T D Y$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\therefore$ The canonical form is $0y_1^2 + y_2^2 + 3y_3^2$

$\therefore$ The nature of the quadratic form is positive semi definite.

Rank $(r) = 2$

Index $(p) = 2$

signature $(s) = 2$

Example: 5:

Reduce the quadratic to canonical form $2x_1x_2 + 2x_1x_3 - 2x_2x_3$ and discuss about its nature.