Unit II

ORTHOGONAL TRASFORMATION OF SYMMETRIC MATRIX OF A DIAGONAL FORM

The transformation $N^T A N = D$ is known as orthogonal transformation or reduction where N is normalized matrix

D is diagonal matrix whos diagonal elements are eigen values of given matrix

Methods through diagonalized a symmetric matrix by Orthogonal Transformation

Step 1:

Find the characteristic equation and eigenvalue and eigen vector. of a matrix A.

Step 2:

Check orthogonal condition

$$x_1^T x_2 = 0, \quad x_2^T x_3 = 0, \quad x_3^T x_1 = 0$$

Step 3:

The normalized eigen vector of $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

then the normalized vector

$$N = \begin{bmatrix} \frac{x_1}{l(x_1)} \\ \frac{x_2}{l(x_2)} \\ \frac{x_3}{l(x_3)} \end{bmatrix}$$

Step 4: Find model matrix $N = [N_1 \; N_2 \; N_3]$

Step 5: Find $N^T A N = D$

Example 1:

Diagonalized the matrix by orthogonal transformation

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Sol

Find characteristic equation, eigenvalue & eigenvector

Characteristic equation

$$\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0$$

c_1 = Sum of the main diagonal elements

$$= 8+7+3 = 18$$

c_2 = Sum of minor of main diagonal elements

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$$= (21-16) + (24-4) + (56-36)$$

$$= 5 + 20 + 20 = 45$$

$$c_3 = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + 6\begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix} + 2\begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}$$

$$= 8(21-16) + 6(-18+8) + 2(24-14)$$

$$= 8(5) + 6(-10) + 2(10)$$

$$= 40 - 60 + 20 = 0$$

C.E

$$\lambda^3 - 18\lambda^2 + 45\lambda + 0 = 0$$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = 3, \quad \lambda_3 = 15$$

45

-3 -15

To find eigen vector

The eigen vector X corresponding to the eigen value λ is given by $(A-\lambda I)X = 0$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case i) $\lambda = 0$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} 8 & -6 & 2 \\ -6 & 7 & -4 \end{matrix}$$ Row must be different

$$\begin{matrix} 8 & \overbrace{-6 \quad 2}^{x_1} & 8 \\ \underbrace{-6 \quad 7}_{x_3} & \underbrace{-4 \quad -6}_{x_2} \end{matrix}$$ Repeat first column

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & 8 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Case ii) $\lambda = 3$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} 5 & -6 & 2 \\ -6 & 4 & -4 \end{matrix}$$

$$\begin{matrix} & x_1 & & \\ 5 & -6 & 2 & 5 \\ -6 & 4 & -4 & -6 \\ x_3 & & x_2 & \end{matrix}$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & 5 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Case iii) $\lambda = 15$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} -7 & -6 & 2 \\ -6 & -8 & -4 \end{matrix}$$

$$\begin{array}{cccc} & & x_1 & \\ -7 & -6 & 2 & -7 \\ -6 & -8 & -4 & -6 \\ & x_3 & & x_2 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & -7 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Step 2

Check orthogonal condition

T.P $X_1^T X_2 = 0$

$$X_1^T X_2 = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$= 2 + 2 - 4 = 0$$

$$X_2^T X_3 = \begin{bmatrix} 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$= 4 - 2 - 2$$

$$= 0$$

$$X_3^T X_1 = \begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$= 2 - 4 + 2$$

$$= 0$$

Step 3

Find normalized matrix

Eigen vector $\quad l(x) = \sqrt{x_1^2 + x_2^2 + x_3^2} \quad$ Normalized vector

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad l(x_1) = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3 \quad N_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \quad l(x_2) \quad N_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad N_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

Step 4.

Model matrix

$$N = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$N^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$N^T A N = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} = D$$