Example : 3

Verify Cayley - Hamilton theorem for $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

and hence find A^{-1} also A^4

Sol

The C.E $\lambda^3 - C_1\lambda^2 + C_2\lambda - C_3 = 0$

D_1 $C_1 = 6$

$C_2 = 9$

$C_3 = 4$

the C.E is $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

To verify, $A^3 - 6A^2 + 9A - 4 = 0$

$$A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = 0 \rightarrow ①$$

Hence verified.

To find A^{-1} in eqn ①

$$A^3A^{-1} - 6A^2A^{-1} + 9AA^{-1} - 4IA^{-1} = 0$$

$$\Rightarrow \quad A^2 - 6A + 9I - 4A^{-1} = 0$$

$$A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & +1 & 3 \end{bmatrix}$$

To find A^4, Premultiply by eqn ①

$$A^4 - 6A^3 + 9A^2 - 4AI = 0$$

$$A^4 = \begin{bmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{bmatrix}$$

4) verify $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ i) A^{-1} ii) A^6

C.E $\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0$

$c_1 = 4$

$c_2 = 6$

$c_3 = 3$

To verify C-H theorem,

$A^3 - 4A^2 + 6A - 3I = 0$

$$A^2 = \begin{bmatrix} 0 & 0 & -3 \\ 5 & 1 & 1 \\ 3 & 0 & 3 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -3 & 0 & -6 \\ 8 & 1 & -2 \\ 6 & 0 & 3 \end{bmatrix}$$

$A^3 - 4A^2 + 6A - 3I = 0$

Hence verified.

To find A^{-1}, premultiply by A^{-1}

$$A^{-1}A^3 - 4A^{-1}A^2 + 6A^{-1}A - 3IA^{-1} = 0$$

$$A^2 - 4A + 6I - 3A^{-1} = 0$$

$$A^{-1} = \frac{1}{3}(A^2 - 4A + 6I)$$

$$= \frac{1}{3}\begin{bmatrix} 2 & 0 & 1 \\ -3 & 3 & -3 \\ -1 & 0 & 1 \end{bmatrix}$$

To find A^6, premultiply by A^3

$$A^3(A^3 - 4A^2 + 6A - 3I) = 0$$

$$A^6 = 4A^5 - 6A^4 + 3A^3 \rightarrow (2)$$

$$A^4 = \begin{bmatrix} -9 & 0 & -9 \\ 8 & 1 & -11 \\ 9 & 0 & 0 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} -18 & 0 & -9 \\ -1 & 1 & -29 \\ 9 & 0 & -9 \end{bmatrix}$$

From (2)

$$A^6 = \begin{bmatrix} -27 & 0 & 0 \\ -28 & 1 & -56 \\ 0 & 0 & 27 \end{bmatrix}$$

Example 5:

Using C.H theorem, verify the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{bmatrix}$

and show simplify the expression

$A^8 - A^7 + 5A^6 - A^5 + A^4 - A^3 + 6A^2 + A - 2I$

Sol.

The C.E is $\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0$

$c_1 = 1$

$c_2 = 5$ $\qquad$ $\therefore$ C.E is

$c_3 = 1$ $\qquad$ $\lambda^3 - \lambda^2 + 5\lambda - 1 = 0$

To verify CH theorem

$$A^3 - A^2 + 5A - I = 0$$

$$A^2 = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & -4 \\ 2 & 6 & -9 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -5 & -12 & 10 \\ -10 & -23 & 16 \\ -13 & -29 & 17 \end{bmatrix}$$

$$A^3 - A^2 + 5A - I = 0$$

The above eqn ① divide by $A^3 - A^2 + 5A - I$

Simplification

$$\begin{array}{r|l} & A^5 + A \\ \hline A^3 - A^2 + 5A - I & A^8 - A^7 + 5A^6 - A^5 + A^4 - A^3 + 6A^2 + A - 2I \\ & A^8 - A^7 + 5A - A^5 \\ \hline & A^4 - A^3 + 6A^2 + A - 2I \\ & A^4 - A^3 + 5A^2 - A \\ \hline & A^2 + 2A - 2I \end{array}$$

$$A^2 + 2A - 2I = \begin{bmatrix} -1 & 2 & -4 \\ 4 & 9 & -12 \\ 8 & 20 & -21 \end{bmatrix}$$