Canonical Form:

The Reduction of an arbitrary quadratic form (by some non-singular linear transformation) to a sum of squares of the unknowns (or) to a form where all coefficient of product of distinct unknowns are zero is a special form of the quadratic form called as a canonical form.

i.e., A quadratic form X^TAX in n unknowns, $x_1, x_2, \ldots, x_n$ can be reduced (via a non-singular linear transformation) to the canonical form $d_1y_1^2 + d_2y_2^2 + \cdots + d_ny_n^2$ where $y_1, y_2, \ldots, y_n$ are the new unknowns, but some of the coefficients $d_1, d_2, \ldots, d_n$ may of course be zeros.

Nature of the quadratic form:

If $Q = X^TAX$ is the quadratic form in 'n' variables $x_1, x_2, \ldots, x_n$. ~~If~~ Then

(i) Rank: Number of non-zero Eigen Values.

(ii) Index: The number of positive square terms in the canonical form.

(iii) Signature: The difference of the number of positive and negative square terms in the canonical form.

(iv) Nature: Q is said to be

a) positive definite – if all the Eigen Values are positive numbers.

b) Negative definite – if all the Eigen Values are negative numbers.

c) positive semi-definite – if all the Eigen Values are positive and atleast one E.V is zero

d) Negative Semi-definite – if all the Eigen Values are negative and atleast one Eigen Value is zero.

e) Indefinite – if the form has both positive and negative Eigen Values.

Problems:

1. Determine the nature of the following quadratic form $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2$

Solution:

The matrix of the quadratic form is

$$Q = \begin{bmatrix} \text{coe. of } x_1^2 & \frac{1}{2}\text{coe. of } x_1x_2 & \frac{1}{2}\text{coe. of } x_1x_3 \\ \frac{1}{2}\text{coe. of } x_1x_2 & \text{coe. of } x_2^2 & \frac{1}{2}\text{coe. of } x_2x_3 \\ \frac{1}{2}\text{coe. of } x_1x_3 & \frac{1}{2}\text{coe. of } x_2x_3 & \text{coe. of } x_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The Eigen Values are 1, 2 and 0.

Since all the Eigen Values are positive and one E.V is zero, the nature of the quadratic form is positive semi-definite.

2. Find the rank, index, signature and nature for the following Canonical forms.

(i) $y_1^2 + 3y_2^2 + 4y_3^2$

(ii) $-y_1^2 + y_2^2 + 4y_3^2$

(iii) $3y_2^2 + 15y_3^2$

(i) Rank = no. of non-zero Eigen Values
= 3.

Index = number of positive square terms = 3.

Signature = Diff b/w positive and negative square terms = 3-0 = 3.

Nature - Positive Definite

(ii) Rank = 3

Index = 2

Signature = 2 - 1 = 1

Nature - Indefinite

(iii) Rank = 2

Index = 2

Signature = 2

Nature - positive Semidefinite.

Reduction of Quadratic form to Canonical form by orthogonal Transformation.

form by orthogonal Transformation.

1) Reduce the given quadratic form $x_1^2+3x_2^2+3x_3^2-2x_2x_3$ into canonical form by using orthogonal transformation. Find its nature.

Solution:

1) Given $Q = x_1^2+3x_2^2+3x_3^2-2x_2x_3$

Now the matrix of the quadratic form is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

H.W

1) $2x_1^2+x_2^2+x_3^2+2x_1x_2-2x_1x_3-4x_2x_3$ $(1,-1,4)$ pg:1.161

2) $x_1^2+2x_2^2+x_3^2-2x_1x_2+2x_2x_3$ $(0,1,3)$ pg: 1.165 and pg: 1.120

) $3x_1^2+5x_2^2+3x_3^2-2x_2x_3+2x_3x_1-2x_1x_2$ $(2,3,6)$ pg: 1.176

Steps:

1) Write the matrix of the given quadratic form
2) To find the characteristic Equation
3) To solve the characteristic Equation
4) To find the Eigenvectors orthogonal to each other.
5) Form Normalized matrix N.
6) Find N^T
7) Find AN
8) Find $D = N^T AN$
9) Canonical Form $[y_1 \ y_2 \ y_3][D]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

2) C.E

The C.E. of a 3x3 matrix is given by

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

$S_1 = 1+3+3 = 7$; $S_2 = 9-1+3+3 = 14$; $S_3 = 8$.

$\therefore$ The C.E is $\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$. ——— (1).

3) Solve C.E

$$\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0.$$

$\lambda = 1$, C.E $= 0$.

$\lambda = 1$ is a root.

$$\begin{array}{c|cccc} 1 & 1 & -7 & 14 & -8 \\ & 0 & 1 & -6 & 8 \\ \hline & 1 & -6 & 8 & \boxed{0} \end{array}$$

$\lambda^2 - 6\lambda + 8 = 0 \Rightarrow \lambda = 2, 4$.

$\therefore$ The Eigen Values are 1, 2 and 4.

4) Eigen Vector

$$(A - \lambda I) X = 0.$$

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- } (2)$$

Case (i): $\lambda = 1$

$$(2) \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x_1 + 0x_2 + 0x_3 = 0$$

$$0x_1 + 2x_2 - 1x_3 = 0$$

$$0x_1 - x_2 + 2x_3 = 0$$

Consider 2^{nd} and 3^{rd} row equations.

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 2 & -1 & 0 & 2 \\ -1 & 2 & 0 & -1 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix}}$$

$$\frac{x_1}{4-1} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$\frac{x_1}{3} = \frac{x_2}{0} = \frac{x_3}{0} \qquad X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

Case (ii): $\lambda = 2$.

$$(2) \Rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

consider first and second row in

$$-x_1 + 0x_2 + 0x_3 = 0$$

$$0x_1 + 1x_2 - 1x_3 = 0$$

$$0x_1 - 1x_2 + 1x_3 = 0$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} 0 & 0 \\ 1 & -1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}}$$

$$\frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Case (iii) $\lambda = 4$

$$(2) \Rightarrow \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x_1 + 0x_2 + 0x_3 = 0$$

$$0x_1 - 1x_2 - 1x_3 = 0$$

$$0x_1 - 1x_2 - 1x_3 = 0$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 0 & 0 & -3 & 0 \\ -1 & -1 & 0 & -1 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} 0 & 0 \\ -1 & -1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 0 & -3 \\ -1 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 0 \\ 0 & -1 \end{vmatrix}}$$

$$\frac{x_1}{0} = \frac{x_2}{-3} = \frac{x_3}{3} \qquad X_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Check : Orthogonal Property

$$X_1^T X_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = [0]$$

$$X_1^T X_3 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = [0]$$

$$X_2^T X_3 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = [0 - 1 + 1] = [0]$$

$\therefore$ The Eigen Vectors satisfy the orthogonal Property.

4) Normalized Modal Matrix, N.

$$l = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{1^2 + 0^2 + 0^2} = \sqrt{1} = 1 \quad \text{for } x_1$$

$$= \sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{2} \quad \text{for } x_2$$

$$= \sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{2} \quad \text{for } x_3$$

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

6) Find N^T

$$N^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

7) Find AN.

$$AN = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

8) Find $D = N^T A N$.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

9) Canonical form :

$$Y^T D Y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_1^2 + 2y_2^2 + 4y_3^2$$

10) Nature :

Index = 3

Rank = 3

Signature = 3 − 0 = 3

Nature is positive Definite.

Extra:

$\cancel{1)\ 6x^2+3y^2+3z^2-4xy-2yz+4zx}$