To find the initial distribution with the no. of females in each class change by the same proportion, consider the equation

$$x + 0.5x + 0.125x = 500 + 500 + 500$$

$$1.625x = 1500$$

$$x = 923$$

age class 1 : $x = 923$

age class 2 : $0.5x = 0.5(923) = 461.5 \simeq 461$

age class 3 : $0.125x = 0.125(923) = 115.37 \simeq 115$

Elastic Deformations:

Stretching of an Elastic Membrane:

Problem:

An Elastic Membrane in the x_1x_2-plane with boundary circle $x_1^2 + x_2^2 = 1$ is stretched so that a point $P:(x_1, x_2)$ goes over into the point $Q:(y_1, y_2)$ given by

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = AX = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In components

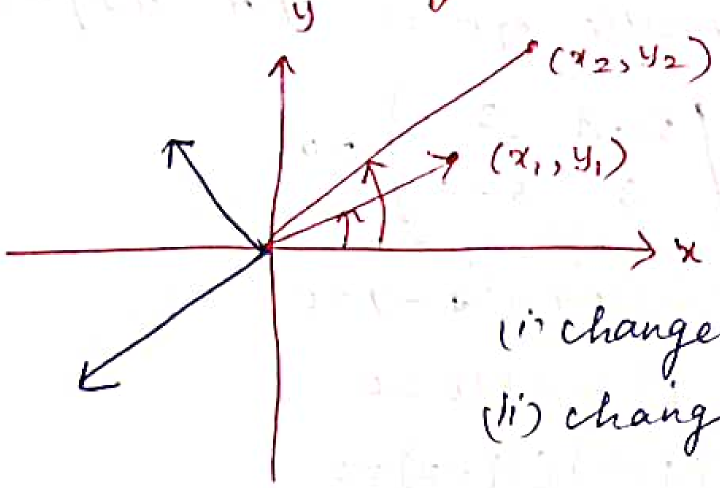
$y_1 = 5x_1 + 3x_2$

$y_2 = 3x_1 + 5x_2$

Find the principal directions, i.e., the directions of the position vector x of P for which the direction of the position vector y of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation

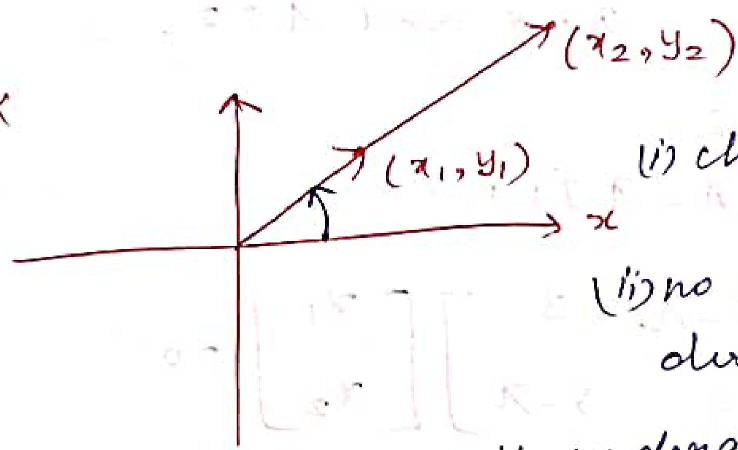
Geometrical Interpretation of Eigen Values and Eigen Vector.

$Y = AX$



(i) change in magnitude
(ii) change in direction.

$Y = AX$



(i) change in magnitude
(ii) no change in direction.

such a special vector where it undergoes change in magnitude but no change in direction is called a Eigen vector.

By what factor or how many times the magnitude changes is the Eigen value.

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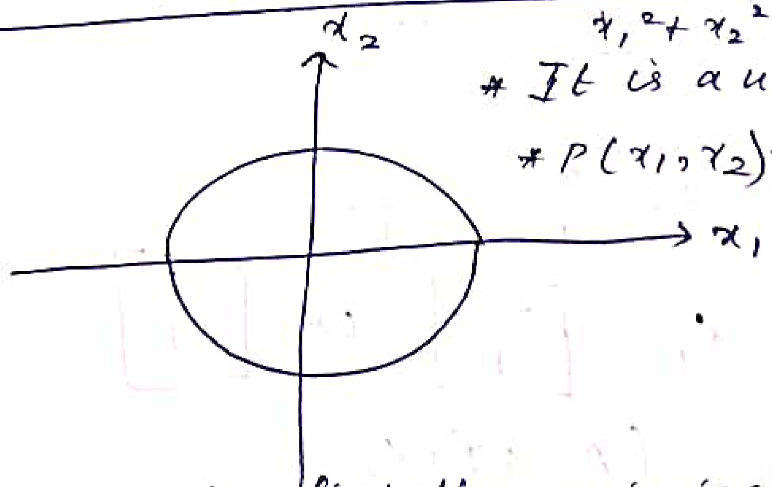
problem:



$x_1^2 + x_2^2 = 1$

* It is a unit circle
* $P(x_1, x_2) \longrightarrow Q(y_1, y_2)$ with the given transformation

* Also we need to find the principal directions, of x to the change y same or opposite.

Given: $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

The characteristic eqn is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)^2 - 9 = 0$$

$$25 - 10\lambda + \lambda^2 - 9 = 0.$$

$$\lambda^2 - 10\lambda + 16 = 0$$

$$(\lambda-8)(\lambda-2) = 0$$

$$\lambda = 8 \text{ and } \lambda = 2.$$

when $\lambda = \cancel{8}\,2$,

$$(A - \lambda I)X_1 = 0.$$

$$\begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$3x_1 + 3x_2 = 0$$ (II eqn is same. avoid)

$$3x_1 = -3x_2$$

$$\frac{x_1}{-3} = \frac{x_2}{3}.$$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{1}$$

$$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

when $\lambda = 8$, $(A - \lambda I)X_2 = 0$

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$-3x_1 + 3x_2 = 0$

$3x_2 = 3x_1$ $\qquad X_2 = \begin{bmatrix}1\\1\end{bmatrix}$

$\frac{x_2}{1} = \frac{x_1}{1}$

for $\lambda = 2$; $X_1 = \begin{bmatrix}-1\\1\end{bmatrix}$ or $\begin{bmatrix}1\\-1\end{bmatrix}$

for $\lambda = 8$; $X_2 = \begin{bmatrix}1\\1\end{bmatrix}$

principal direction

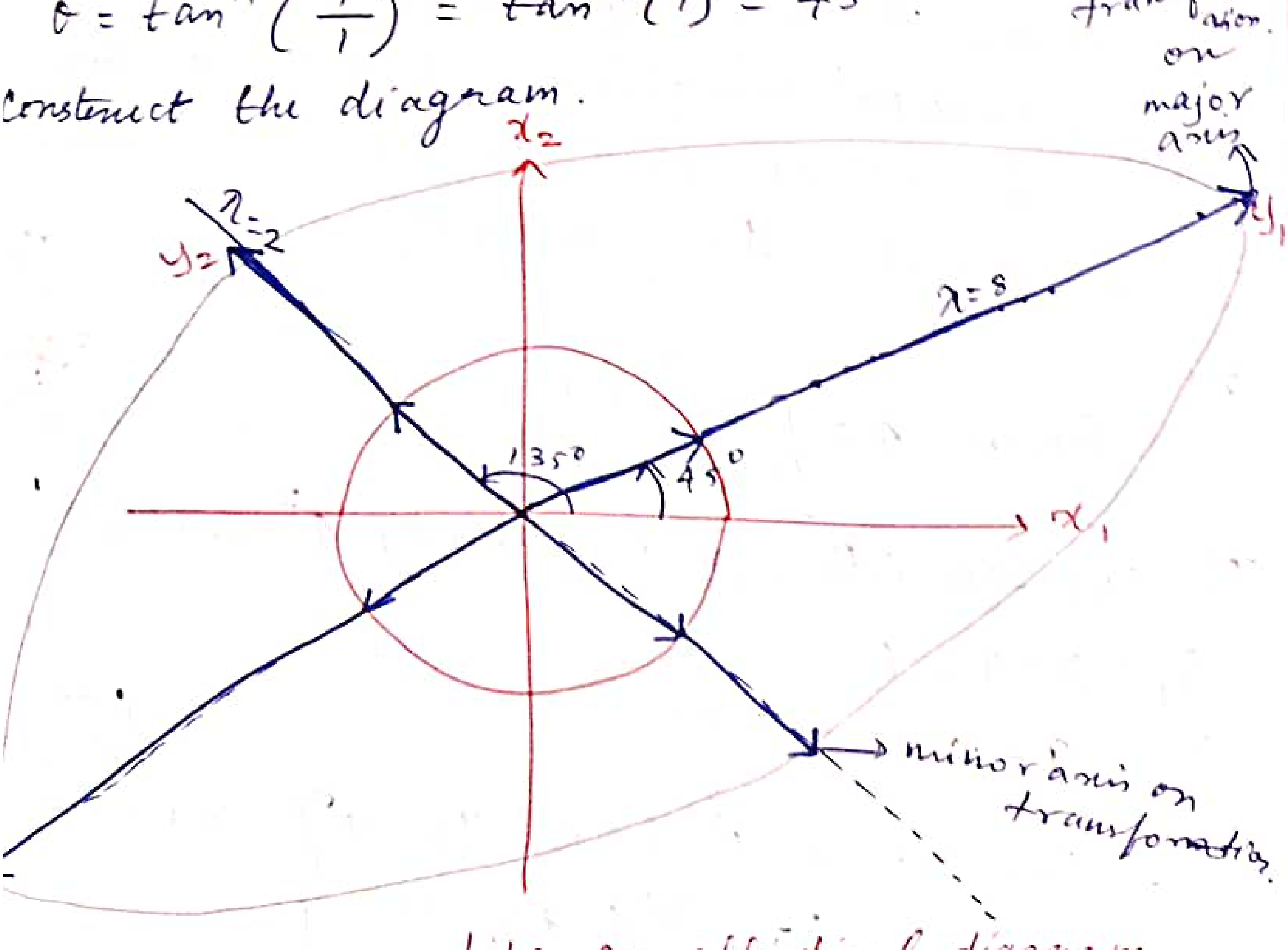
$\theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$ for X_1

$\theta = \tan^{-1}\left(\frac{1}{-1}\right) = \tan^{-1}(-1) = 135^\circ$

$\theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$ for X_2

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = 45^\circ$.

Construct the diagram.



Like an elliptical diagram.

after transformation to $Q(y_1, y_2)$.

The circle has transformed to ellipse general $\frac{x^2}{a^2}+\frac{y^2}{b^2}$

Major axis y_1

Minor axis y_2

Here, $\frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} = 1$

Now to find a and b

$2a = 16$ | $2b = 4$

$a = 8$ | $b = 2$.

$\therefore$ The equation of the ellipse is $\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$

This is the final deformation of the circle into an ellipse, $\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$.

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(2) Given A in a deformation $Y = AX$.

Find the principal directions (Eigenvectors) and corresponding factors of extension or contraction $\Rightarrow$ Eigen values. Show the details. $A = \begin{bmatrix} 3 & 1.5 \\ 1.5 & 3 \end{bmatrix}$

Soln:

Given: $A = \begin{bmatrix} 3 & 1.5 \\ 1.5 & 3 \end{bmatrix}$

The characteristic eqn is given by $\lambda^2 - S_1\lambda + S_2 = 0$

$S_1 = 3 + 3 = 6$

$S_2 = 9 - 2.25 = 6.75$

$\therefore$ The C.E is $\lambda^2 - 6\lambda + 6.75 = 0$

$(\lambda - 1.5)(\lambda - 4.5) = 0$

$\lambda = 1.5, 4.5$

$1.5 \times 1.5 = 2.25$ $\quad$ 6.75 $\quad$ $-1.5 \;\; -4.5$

$$|A-\lambda I|=0$$

$$\begin{vmatrix} 3-\lambda & 1.5 \\ 1.5 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)^2-(1.5)^2=0$$

$$(3-\lambda)^2 = 1.5^2$$

$$3-\lambda = \pm 1.5$$

$$\lambda = 3 \pm 1.5$$

$$\lambda = 4.5 \text{ or } 1.5.$$

when $\lambda = 1.5$

$$(A-\lambda I)X_1 = 0$$

$$\begin{bmatrix} 3-\lambda & 1.5 \\ 1.5 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1.5 & 1.5 \\ 1.5 & 1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$1.5x_1 + 1.5x_2 = 0$$

$$1.5x_1 = -1.5x_2$$

$$\frac{x_1}{-1.5} = \frac{x_2}{1.5}$$

$$\frac{x_1}{-1} = \frac{x_2}{1}$$

$$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

when $\lambda = 4.5$

$$\begin{bmatrix} -1.5 & 1.5 \\ 1.5 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-1.5x_1 + 1.5x_2 = 0$$

$$1.5x_1 - 1.5x_2 = 0$$

$$-1.5x_1 = -1.5x_2$$

$$\frac{x_1}{1} = \frac{x_2}{1} \qquad X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

∴ The Eigen values are $\lambda = 1.5$ $\lambda = 4.5$

The Eigen Vectors are $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Principal direction:

$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\tan\theta = \frac{x_2}{x_1}$

$\theta = \tan^{-1}\left(\frac{-1}{1}\right)$

$= 135^\circ$.

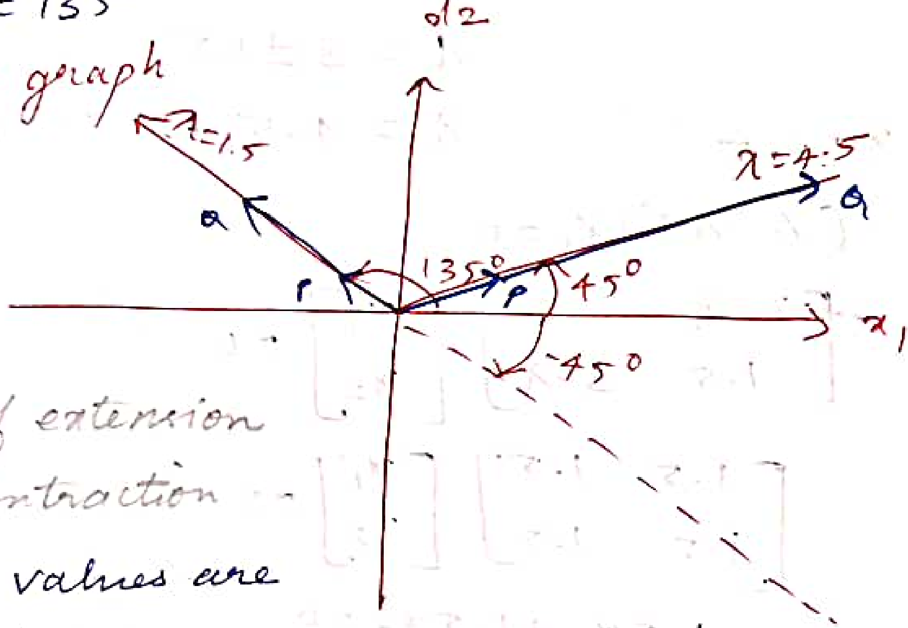
$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\tan\theta = \frac{x_2}{x_1} = \frac{1}{1}$

$\theta = \tan^{-1}(1)$

$= 45^\circ$.

plot in graph



factors of extension or contraction

since λ values are +ve, it is an extension.

(i) The principal direction for the given transformation is -45° or -135° and its extension factor is 1.5.

(ii) The principal direction is 45° and extension factor is 4.5.

3) Given $A = \begin{bmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 2 \end{bmatrix}$ in a deformation $Y = AX$, find the principal directions and corresponding factors of extension or contraction.

Solution:

We are looking for vectors X such that $Y = \lambda X$. Since $Y = AX$, we get $AX = \lambda X$, Therefore, we have to find the Eigen Value and the Eigen Vector for the matrix A. The characteristic eqn is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 7-\lambda & \sqrt{6} \\ \sqrt{6} & 2-\lambda \end{vmatrix} = 0.$$

$$(7-\lambda)(2-\lambda) - (\sqrt{6})^2 = 6$$

$$14 - 7\lambda - 2\lambda + \lambda^2 - 6 = 0$$

$$\lambda^2 - 9\lambda + 8 = 0.$$

$$\Rightarrow (\lambda - 1)(\lambda - 8) = 0$$

$$\lambda = 1 \text{ or } 8.$$

for $\lambda = 1$; The Eigen Vector of A is given by

$$(A - \lambda I)X = 0$$

$$\Rightarrow \begin{bmatrix} 7-\lambda & \sqrt{6} \\ \sqrt{6} & (2-\lambda) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$$\cancel{(7-\lambda)(2-\lambda)}$$

$$\cancel{(7-\lambda)x_1 + \sqrt{6}x_2 = 0}$$

$$\cancel{\sqrt{6}x_1 + (2-\lambda)x_2 = 0}.$$

$$\begin{bmatrix} 6 & \sqrt{6} \\ \sqrt{6} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$$6x_1 + \sqrt{6}x_2 = 0$$

$$\sqrt{6}x_1 + x_2 = 0$$

$$6x_1 = -\sqrt{6}\,x_2$$

$$\Rightarrow \frac{x_1}{-\sqrt{6}} = \frac{x_2}{6}$$

$$\frac{x_1}{-1} = \frac{x_2}{\sqrt{6}} \quad \text{or} \quad \frac{\cancel{6}^{\sqrt{6}}\,x_1}{-\sqrt{\cancel{6}}} = \frac{x_2}{1}$$

$$\frac{x_1}{\left(-\frac{1}{\sqrt{6}}\right)} = \frac{x_2}{1}$$

$$X_1 = \begin{bmatrix} -1 \\ \sqrt{6} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -1/\sqrt{6} \\ 1 \end{bmatrix}$$

This vector makes angle with positive x_1 direction which is $\tan^{-1}\left(\frac{x_2}{x_1}\right) = \tan^{-1}\left(\frac{1}{(-1/\sqrt{6})}\right)$

$$= \tan^{-1}(-\sqrt{6})$$

$$= -\tan^{-1}(\sqrt{6})$$

to eliminate -ve

$$= 180^\circ - \tan^{-1}(\sqrt{6})$$

$$= 112.2$$

$\therefore$ This vector makes 112.2° with positive x_1 direction.

case (ii) $\lambda_2 = 8$

$$\begin{bmatrix} -1 & \sqrt{6} \\ \sqrt{6} & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-x_1 + \sqrt{6}\,x_2 = 0$$

$$\sqrt{6}\,x_1 - \sqrt{6}\,x_2 = 0$$

$$x_1 = \sqrt{6}\,x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{(1/\sqrt{6})} \qquad X_1 = \begin{bmatrix} 1 \\ 1/\sqrt{6} \end{bmatrix}$$

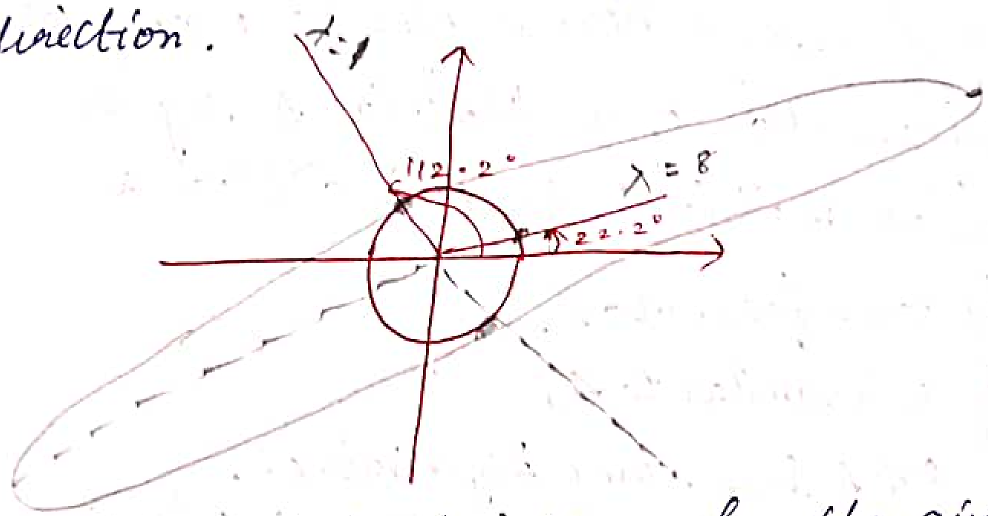
This vector makes an angle with positive x_1 direction is

$$\tan^{-1}\left(\frac{x_2}{x_1}\right) = \tan^{-1}\left(\frac{1/\sqrt{6}}{1}\right)$$

$$= \tan^{-1}\left(\frac{1}{\sqrt{6}}\right)$$

$$= 22.2$$

This vector makes 22.2° ~~angles~~ with positive x_1 direction.



∴ The principal direction for the given transformation is -22.2° or 112.2° and its extension factor is 1.

and The principal direction is 22.2° and its extension factor is 8.