Cayley Hamilton Theorem

Theorem: (Statement).

Every square matrix satisfies its own characteristic equation.

Applications:

* This method is a practical method for computation of the inverse of the large matrices.
* This method is useful to find the inverse of a square matrix A in terms of (n-1) powers of matrix A.
* The higher positive integral powers of A can also be calculated using this theorem.

(1) Verify Cayley Hamilton Theorem for $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and also find A^4.

Solution: Given $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

The characteristic eqn of A is given by

$$\lambda^2 - S_1\lambda + S_2 = 0.$$

S_1 = sum of the diagonal elements

$= 1 - 1 = 0$

$S_2 = |A|$

$= \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1 - 4 = -5$

$\therefore$ The characteristic equation is

$$\lambda^2 - 0 - 5 = 0.$$

$$\lambda^2 - 5 = 0$$

Verification:

check $A^2 - 5I = 0$

$$A^2 = A \times A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+4 & 2-2 \\ 2-2 & 4+1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$5I = 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$A^2 - 5I = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} =$$

Hence, Cayley-Hamilton Theorem is verified.

To find A^4:

We know that $A^2 - 5I = 0$

$$\Rightarrow A^2 = 5I$$

$\times$ by A^2

$$A^4 = (5I)A^2$$

$$A^4 = (5I)(5I)$$

$$A^4 = 25(I)$$

$$= 25\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow A^4 = \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$$

hw

2. Verify Cayley Hamilton Theorem for $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and also find A^{-1}.

Soln:

The characteristic eqn is given by

$$\lambda^2 - S_1\lambda + S_2 = 0$$

S_1 = sum of the diagonal elements

$= 1+3 = 4$

$$S_2 = |A| = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = 3-8 = -5.$$

∴ The characteristic eqn is $\lambda^2 - 4\lambda - 5 = 0$.

Verification:

check $A^2 - 4A - 5I = 0$.

$$A^2 = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1+8 & 4+12 \\ 2+6 & 8+9 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 16 \\ 8 & 17 \end{pmatrix}$$

$$4A = 4\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 16 \\ 8 & 12 \end{pmatrix}$$

$$5I = 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

L.H.S

$$= \begin{pmatrix} 9 & 16 \\ 8 & 17 \end{pmatrix} - \begin{pmatrix} 4 & 16 \\ 8 & 12 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}.$$

$$= \begin{pmatrix} 9-4-5 & 16-16-0 \\ 8-8-0 & 17-12-5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 = \text{RHS}$$

Hence C-H Theorem is verified

To find A^{-1}:

$$A^2 - 4A - 5I = 0$$

$$A^2 - 4A = 5I$$

$$\times A^{-1} \Rightarrow 5IA^{-1} = (A^2 - 4A)A^{-1}$$

$$5A^{-1} = A^2 \cdot A^{-1} - 4A \cdot A^{-1}$$

$$5A^{-1} = A - 4I$$

$$5A^{-1} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$5A^{-1} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1-4 & 4-0 \\ 2-0 & 3-4 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}$$

$$\Rightarrow \boxed{A^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}}$$

3. Verify Cayley-Hamilton Theorem for $A = \begin{bmatrix} 0 & 0 & 2 \\ 2 & 1 & 0 \\ -1 & -1 & 3 \end{bmatrix}$ and find A^{-1}.

Soln: The characteristic equation is given by

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

S_1 = sum of the diagonal elements $= 0+1+3 = 4$

S_2 = sum of the minors of the diagonal elements

$$= \begin{vmatrix} 1 & 0 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix}$$

$$= 3 + 2 + 0 = 5$$

$$S_3 = |A| = 2(-2+1) = 2(-1) = -2$$

$\therefore$ The characteristic eqn is

$$\lambda^3 - 4\lambda^2 + 5\lambda + 2 = 0$$

Verification:

check $A^3 - 4A^2 + 5A + 2I = 0$

$$A^2 = \begin{pmatrix} 0 & 0 & 2 \\ 2 & 1 & 0 \\ -1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 & 2 \\ 2 & 1 & 0 \\ -1 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0+0-2 & 0+0-2 & 0+0+6 \\ 0+2+0 & 0+1+0 & 4+0+0 \\ 0-2-3 & 0-1-3 & -2+0+9 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -2 & 6 \\ 2 & 1 & 4 \\ -5 & -4 & 7 \end{pmatrix} ; \; 4A^2 = 4\begin{pmatrix} -2 & -2 & 6 \\ 2 & 1 & 4 \\ -5 & -4 & 7 \end{pmatrix} = \begin{pmatrix} -8 & \ldots \\ 8 & \ldots \\ -20 & -1\ldots \end{pmatrix}$$

$$A^3 = A^2 . A = \begin{pmatrix} -2 & -2 & 6 \\ 2 & 1 & 4 \\ -5 & -4 & 7 \end{pmatrix}\begin{pmatrix} 0 & 0 & 2 \\ 2 & 1 & 0 \\ -1 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -4-6 & -2-6 & -4+18 \\ 2-4 & 1-4 & 4+12 \\ -8-7 & -4-7 & -10+21 \end{pmatrix}$$

$$= \begin{pmatrix} -10 & -8 & 14 \\ -2 & -3 & 16 \\ -15 & -11 & 11 \end{pmatrix}$$

$$5A = 5\begin{pmatrix} 0 & 0 & 2 \\ 2 & 1 & 0 \\ -1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 10 \\ 10 & 5 & 0 \\ -5 & -5 & 15 \end{pmatrix}$$

$$2I = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L.H.S = \begin{pmatrix} -10 & -8 & 14 \\ -2 & -3 & 16 \\ -15 & -11 & 11 \end{pmatrix} - \begin{pmatrix} -8 & -8 & 24 \\ 8 & 4 & 16 \\ -20 & -16 & 28 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 10 \\ 10 & 5 & 0 \\ -5 & -5 & 15 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$= 0$

$= R.H.S$

Hence Cayley-Hamilton Theorem is verified.

To find A^{-1}:

$$A^3 - 4A^2 + 5A + 2I = 0.$$

$$\Rightarrow 2I = -A^3 + 4A^2 - 5A$$

$\times A^{-1}$

$$2IA^{-1} = -A^3 \cdot A^{-1} + 4A^2A^{-1} - 5A \cdot A^{-1}$$

$$2A^{-1} = -A^2 + 4A - 5I$$

$$2A^{-1} = -\begin{pmatrix} -2 & -2 & 6 \\ 2 & 1 & 4 \\ -5 & -4 & 7 \end{pmatrix} + 4\begin{pmatrix} 0 & 0 & 2 \\ 2 & 1 & 0 \\ -1 & -1 & 3 \end{pmatrix} - \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2+0-5 & 2+0+0 & -6+8+0 \\ -2+8-0 & -1+4-5 & -4+0-0 \\ 5-4-0 & 4-4+0 & -7+12-5 \end{pmatrix}$$

$$2A^{-1} = \begin{pmatrix} -3 & 2 & 2 \\ 6 & -2 & -4 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{2}\begin{pmatrix} -3 & 2 & 2 \\ 6 & -2 & -4 \\ 1 & 0 & 0 \end{pmatrix}$$

hw

4) Verify C-H Theorem and find A^4 and A^{-1} for

$$A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

Solution: The characteristic eqn is given by

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

S_1 = sum of the diagonal elements $= 2+2+2 = 6$

S_2 = sum of the minors of the diagonal elements

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= 3 + 2 + 3 = 8$$

$$S_3 = |A| = 2(4-1) + 1(-2+1) + 2(1-2)$$

$$= 2(3) + 1(-1) + 2(-1) = 6 - 1 - 2 = 3$$

$\therefore$ The characteristic eqn is given by

$$\lambda^3-6\lambda^2+8\lambda-3=0$$

Verification:

check $A^3-6A^2+8A-3I=0$.

$$A^2=A.A=\begin{pmatrix}2&-1&2\\-1&2&-1\\1&-1&2\end{pmatrix}\begin{pmatrix}2&-1&2\\-1&2&-1\\1&-1&2\end{pmatrix}=\begin{pmatrix}7&-6&9\\-5&6&-6\\5&-5&7\end{pmatrix}$$

$$A^3=\begin{pmatrix}7&-6&9\\-5&6&-6\\5&-5&7\end{pmatrix}\begin{pmatrix}2&-1&2\\-1&2&-1\\1&-1&2\end{pmatrix}=\begin{pmatrix}29&-28&38\\-22&23&-28\\22&-22&29\end{pmatrix}$$

$$L.H.S=\begin{pmatrix}29&-28&38\\-22&23&-28\\22&-22&29\end{pmatrix}-6\begin{pmatrix}7&-6&9\\-5&6&-6\\5&-5&7\end{pmatrix}+8\begin{pmatrix}2&-1&2\\-1&2&-1\\1&-1&2\end{pmatrix}-3\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}29-42+16-3&-28+36-8-0&38-54+16-0\\-22+\overset{30}{\cancel{42}}-8-0&23-36+16-3&-28+36-8+0\\22-30+8+0&-22+30-8-0&29-42+16-3\end{pmatrix}$$

$$=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}=0=R.H.S$$

Hence C-H Theorem is verified.

To find A^{-1}.

$$A^3-6A^2+8A-3I=0.$$

$$\Rightarrow 3I=-A^3+6A^2-8A.$$

$\times A^{-1}$ $\quad 3A^{-1}=-A^2+6A-8I$

$$3A^{-1}=-\begin{bmatrix}7&-6&9\\-5&6&-6\\5&-5&7\end{bmatrix}+6\begin{bmatrix}2&-1&2\\-1&2&-1\\1&-1&2\end{bmatrix}-8\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$3A^{-1} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} + \begin{bmatrix} 12 & -6 & 12 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

To find A^4:

$$A^3 - 6A^2 + 8A - 3I = 0.$$

$$\times A \quad A^4 - 6A^3 + 8A^2 - 3A = 0.$$

$$A^4 = -8\begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} + 6\begin{pmatrix} 29 & -28 & 38 \\ 22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix} - 3\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 174 & -168 & 228 \\ -132 & 138 & -168 \\ 132 & -132 & 174 \end{pmatrix} - \begin{pmatrix} 56 & -48 & 72 \\ -40 & 48 & -48 \\ 40 & -40 & 56 \end{pmatrix} - \begin{pmatrix} 6 & -3 & 6 \\ -3 & 6 & -3 \\ 3 & -3 & 6 \end{pmatrix}$$

$$\Rightarrow A^4 = \begin{pmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ -89 & -95 & 124 \end{pmatrix}$$

5. Verify C-H Theorem for $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and find A^{-1}. Also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10$ as a linear Polynomial in A.

Soln:

The characteristic eqn is given by

$$\lambda^2 - S_1\lambda + S_2 = 0.$$

$S_1 = 1 + 3 = 4 \quad ; \quad S_2 = 3 - 8 = -5$

$\therefore \lambda^2 - 4\lambda - 5 = 0$ is the C.E of A.

check:

By Cayley Hamilton Theorem, $A^2 - 4A - 5I = 0$

L.H.S.

$$= \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - 4\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 16 \\ 8 & 17 \end{pmatrix} - \begin{pmatrix} 4 & 16 \\ 8 & 12 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 9-4-5 & 16-16-0 \\ 8-8-0 & 17-12-5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 = \text{R.H.S.}$$

Hence C-H theorem is verified —— (*)

To find A^{-1},

$$A^2 - 4A - 5I = 0$$

$$5I = A^2 - 4A$$

$$\times A^{-1} \quad 5A^{-1} = A - 4I$$

$$= \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}$$

To find the linear Polynomial

$$
\begin{array}{r|l}
 & A^3-2A+3 \\
\hline
A^2-4A-5I & A^5-4A^4-7A^3+11A^2-A-10I \\
 & A^5-4A^4-5A^3 \\
 & (-)\quad(+)\qquad(+) \\
\hline
 & -2A^3+11A^2-A \\
 & -2A^3+8A^2+10A \\
 & (+)\quad(-)\qquad(-) \\
\hline
 & 3A^2-11A-10I \\
 & 3A^2-12A-15I \\
 & (-)\quad(+)\qquad(+) \\
\hline
 & A+5I
\end{array}
$$

$\therefore A^5-4A^4-7A^3+11A^2-A-10I$

$= (A^2-4A-5I)(A^3-2A+3) + A+5I$

Using (*), we have $A^2-4A-5I = 0$

$\therefore \quad = (0)(A^3-2A+3) + A+5I$

$= A+5I$, which is the linear polynomial in A.

b) Given that $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$. Compute the value of $A^6-5A^5+8A^4-2A^3-9A^2+31A-36I$ using C-H Theorem.

Soln:

The C.E. of the matrix A is given by

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

$S_1 = 1+1+1 = 3$

$S_2 = (1-1)+(1-3)+(1-0) = 0-2+1 = -1$

$S_3 = |A| = 1(0)-0+3(-2-1) = -9$

$\therefore$ The C.E is $\lambda^3-3\lambda^2-\lambda+9 = 0.$

By cayley-Hamilton Theorem:

$$A^3 - 3A^2 - A + 9I = 0 \quad \text{—— } (*)$$

$$\begin{array}{r|l}
 & A^3 - 2A^2 + 3A - 4 \\
\hline
A^3 - 3A^2 - A + 9I & A^6 - 5A^5 + 8A^4 - 2A^3 - 9A^2 + 31A - 36I \\
 & A^6 - 3A^5 - A^4 + 9A^3 \\
 & (-) \quad (+) \quad (+) \quad (-) \\
\hline
 & -2A^5 + 9A^4 - 11A^3 - 9A^2 \\
 & -2A^5 + 6A^4 + 2A^3 - 18A^2 \\
 & (+) \quad (-) \quad (-) \quad (+) \\
\hline
 & 3A^4 - 13A^3 + 9A^2 + 31A \\
 & 3A^4 - 9A^3 - 3A^2 + 27A \\
 & - \quad (+) \quad (+) \quad (-) \\
\hline
 & 4A^3 + 12A^2 + 4A - 36I \\
 & 4A^3 + 12A^2 + 4A - 36I \\
\hline
 & 0
\end{array}$$

$\therefore \; A^6 - 5A^5 + 8A^4 - 2A^3 - 9A^2 + 31A - 36I$

$= (A^3 - 3A^2 - A + 9I)(A^3 - 2A^2 + 3A - 4)$

$= (0)(A^3 - 2A^2 + 3A - 4) \quad (\text{by } (*))$

$= 0$

7) Find the c.E. of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence compute A^{-1}. Also find the matrix represented by $A^8 - 5A^7 + 7A^6 + 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

Soln:

The C.E. is given by

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

$S_1 = 2+1+2 = 5$

$S_2 = 2+3+2 = 7$

$S_3 = |A| = 1(4-1) = 3$

$\therefore$ The C.E is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0.$

By C-H Theorem, $A^3 - 5A^2 + 7A - 3I = 0$. — (*)

To find A^{-1}.

$$3I = A^3 - 5A^2 + 7A \quad \text{— from (*)}$$

$\times A^{-1}$ $\quad 3A^{-1} = A^2 - 5A + 7I$

$$= \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 10 & 5 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 10 \end{pmatrix} + \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} - \begin{pmatrix} 10 & 5 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 10 \end{pmatrix} + \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$

$$A^5 + A$$

$$A^3 - 5A^2 + 7A - 3I \;\big)\; A^8 - 5A^7 + 7A^6 + 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$A^8 - 5A^7 + 7A^6 - 3A^5$$

$$A^4 - 5A^3 + 8A^2 - 2A + I$$

$$A^4 - 5A^3 + 7A^2 - 3A$$

$$A^2 - A + I$$

$\therefore$ Given $= \left[\dfrac{(A^3-5A^2+7A-3I)}{(A^5+A)}\right] f(A^2+A+I)$

$= A^2+A+I$ (from ⊛)

$= \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Given expression $= \begin{pmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{pmatrix}$ //

CW

8) Verify Cayley-Hamilton Theorem for

$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$. Find A^{-1} and A^4.

Soln.

The C.E is given by $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

$S_1 = 1+1-2 = 0$

$S_2 = -2-1+1 = -3+1 = -2$

$S_3 = |A| = 1(-2) - 1(-1) = -2+1 = -1$

$\therefore$ The C.E is $\lambda^3 - 2\lambda + 1 = 0$.

Verification:

check: $A^3 - 2A + I = 0$ ——— ⊛

L.H.S

$A^2 = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 5 & 1 & -3 \\ -1 & 0 & 3 \end{pmatrix}$

$$A^3 = A^2 \cdot A = \begin{pmatrix} 0 & 0 & 1 \\ 5 & 1 & -3 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -2 \\ 4 & 1 & 2 \\ 2 & 0 & -5 \end{pmatrix}$$

$$L.H.S = \begin{pmatrix} 1 & 0 & -2 \\ 4 & 1 & 2 \\ 2 & 0 & -5 \end{pmatrix} - \begin{pmatrix} 2 & 0 & -2 \\ 4 & 2 & 2 \\ 2 & 0 & -4 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$= 0 = R.H.S.$$

Hence, C-H Theorem is verified.

<u>To find A^{-1} and A^4 :</u>

A^4:

Consider (*) and $\times$ by A, $\Rightarrow A^4 - 2A^2 + A = 0$

$$A^4 = 2A^2 - A$$

$$= \begin{pmatrix} 0 & 0 & 2 \\ 10 & 2 & -6 \\ -2 & 0 & 6 \end{pmatrix} - \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} -1 & 0 & 3 \\ 8 & 1 & -7 \\ -3 & 0 & 8 \end{pmatrix} //$$

A^{-1}

Consider (*) and $\times$ by A^{-1}

$$A^{-1} = A^2 + 2I = \begin{pmatrix} 0 & 0 & 2 \\ 10 & 2 & -6 \\ -2 & 0 & 6 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ -5 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix} //$$