Orthogonal transformation of a symmetric matrix to diagonal form:

Transforming the given matrix A into a diagonal matrix D by means of the transformation $N^T A N = D$ is known as orthogonal transformation or orthogonal reduction.

Problems:

① Diagonalize the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ by means of an orthogonal transformation?

Soln:

$$A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$$

$$\lambda^3 - C_1\lambda^2 + C_2\lambda - C_3 = 0.$$

$$C_1 = 8 + 7 + 3 = 18$$

$$C_2 = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 45$$

$$C_3 = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} = 0$$

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\boxed{\lambda = 0, 3, 15}$$

$$(A-\lambda I)X = 0$$

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case (i) : $\lambda = 0$

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{matrix} 8 & -6 & 2 & 8 \\ -6 & 7 & -4 & -6 \end{matrix}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Case (ii) : $\lambda = 3$

$$\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{matrix} 5 & -6 & 2 & 5 \\ -6 & 4 & -4 & -6 \end{matrix}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

Case (iii) : $\lambda = 15$

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$

(7)

$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Hence the modal matrix is,

$$M = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

The normalised matrix is,

$$N = \begin{pmatrix} 1/\sqrt{1^2+2^2+2^2} & 2/\sqrt{2^2+1^2+(-2)^2} & 2/\sqrt{2^2+(-2)^2+1^2} \\ 2/\sqrt{1^2+2^2+2^2} & 1/\sqrt{2^2+1^2+(-2)^2} & -2/\sqrt{2^2+(-2)^2+1^2} \\ 2/\sqrt{1^2+2^2+2^2} & -2/\sqrt{2^2+1^2+(-2)^2} & 1/\sqrt{2^2+(-2)^2+1^2} \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}$$

$$N = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$N^T = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$N^T A N = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix} = D \Rightarrow \boxed{N^T A N = D}$$

(2) Diagonalize the matrix $A = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix}$ by means of an orthogonal transformation.

Soln:

$$\lambda = -2,\ 6,\ 6$$

$$X = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$N^T A N = D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

(3) Diagonalize the matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$ by means of an orthogonal transformation.

Soln: $\lambda = -1, 1, 4$

(4) Reduce the matrix $\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ to diagonal form.

Soln:

$$\lambda = 1, 1, 4$$