Problems :

(1) Reduce the Quadratic form $6x^2+3y^2+3z^2-4xy-2yz+4zx$ into Canonical form by an orthogonal transformation. Discuss its nature.

Solution:

$$Q = 6x^2+3y^2+3z^2-4xy-2yz+4zx$$

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$\lambda^3 - C_1\lambda^2 + C_2\lambda - C_3 = 0 \rightarrow (1)$$

$$C_1 = 6+3+3 = 12$$

$$C_2 = \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 36$$

$$C_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 32$$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0.$$

$$\boxed{\lambda = 2, 2, 8}$$

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case (i) : $\lambda = 8$

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Case (ii) : $\lambda = 2$

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

All the three rows are equal

$$4x_1 - 2x_2 + 2x_3 = 0.$$

Put $x_1 = 0$, $\quad -2x_2 = -2x_3$

$$\frac{x_2}{1} = \frac{x_3}{1}$$

$$\therefore X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

The given matrix is a symmetric matrix. In this matrix, the third eigen ~~vecto~~ vector X_3 is orthogonal to x_1 & x_2

$$X_3^T X_1 = 0 \Rightarrow (a \;\; b \;\; c)\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 0 \Rightarrow a - b + c = 0 \rightarrow (i)$$

$$X_3^T X_2 = 0 \Rightarrow (a \;\; b \;\; c)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 \Rightarrow 0a + b + c = 0 \rightarrow (ii)$$

Solving (i) & (ii),

$$X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

The modal matrix is,

$$M = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

The normalized matrix is,

$$N = \begin{pmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{pmatrix}$$

$$N^T A N = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D$$

Canonical form :

$$Y^T D Y = (y_1 \quad y_2 \quad y_3) \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= 8y_1^2 + 2y_2^2 + 2y_3^2$$

Index (s) = No. of positive square terms = 3

Rank (r) = No. of non-zero eigen values = 3

Signature $= 2s - r = 6 - 3 = 3$

The quadratic form is positive definite.